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Electromagnetic Field Theory without Divergence Problems 1. The Born Legacy[1,2]

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Born's quest for the elusive divergence problem-free quantum theory of electromagnetism led to the important discovery of the nonlinear Maxwell–Born–Infeld equations for the classical electromagnetic fields, the sources of which are classical point charges in motion. The law of motion for these point charges has however been missing, because the Lorentz self-force in the relativistic Newtonian (formal) law of motion is ill-defined in magnitude and direction. In the present paper it is shown that a relativistic Hamilton–Jacobi type law of point charge motion can be consistently coupled with the nonlinear Maxwell–Born–Infeld field equations to obtain a well-defined relativistic classical electrodynamics with point charges. Curiously, while the point charges are spinless, the Pauli principle for bosons can be incorporated. Born's reasoning for calculating the value of his aether constant is re-assessed and found to be inconclusive.

KEY WORDS: *Spacetime*: special and general relativity; *Electromagnetism*: electromagnetic fields; point charges; *Determinism*: Maxwell–Born–Infeld field equations; Hamilton–Jacobi law of motion; *Permutability*: configuration space; Pauli principle.

1. INTRODUCTION

> "Earlier experience with classical electron theory provided a warning that a point electron will have infinite electromagnetic self-mass;... Disappointingly this problem appeared with even greater severity in the early days of quantum field theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day."
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> Steven Weinberg (ref. 102, p. 31)

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[2] Part I of two parts in honor of Elliott H. Lieb's 70th birthday.

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The making of quantum electrodynamics (QED) is generally recognized as one of the biggest success stories of 20th century physics.[(95)] The agreement of the measured values of the Lamb shift and the electron's *g* factor with those computed perturbatively in QED is nothing less than spectacular.[(35)] Yet, no physicist can be too happy about this feat, for QED's precision rests entirely on truncation. Without mathematical infrared (IR) and ultraviolet (UV) cutoffs QED is divergent at very large and very small length scales, and while renormalization techniques allow these cutoffs to be removed in each order of perturbation theory,[(100)] the resulting series is still divergent[(39)] and must be truncated. Many experts have voiced their dissatisfaction with QED in this regard.[4] It is a difficult open question whether this state of affairs in electromagnetic theory can be overcome.

For some three-plus decades it seemed that QED's divergence problems were merely a temporary nuisance that renormalization techniques would make go away eventually if we just keep working at it hard enough,[(48)] but the likelihood for this to come true seems very remote by now. While efforts to give mathematical sense to renormalized QED continue (e.g., refs. 33 and 34), and progress is made in the rigorous control of non-perturbative low-energy (no pair creation/annihilation) approximations[5] (e.g., refs. 4, 70, and 71), many physicists seem to have resigned to the view that QED, and for that matter also the quantum field theory (QFT) of the standard model of the electromagnetic + weak + strong interactions, has to be relegated from the status of "fundamental" to that of "effective."[(102)] As effective theories, they may well require cutoffs[6] to mimic the hypothetically regularizing effects of the omitted high-energy physics. By high-energy physics one means the realm of the truly fundamental theory, which aims at the unification of all (known) interactions, gravity and the structure of spacetime included. It may sound fantastic that

[4] While R. Penrose files QED in his "SUPERB" category for its numerical precision, he remarks that "[t]he theory as a whole does not have the compelling elegance or consistency of the earlier SUPERB theories,..." (p. 153 in ref. 84). Peierls is more specific and insists that "the use, as basic principle, of a semiconvergent series is unsatisfactory" (p. 92 in ref. 83). For Jost this suffices to declare that he "cannot call QED a theory yet" (p. 93 in ref. 61), and Dirac's and Landau's radical opposition to QED in their later career is well documented in ref. 95. The introductory quotation taken from[(102)] speaks for itself.

[5] The forthcoming monograph[(96)] summarizes the current state of affairs.

[6] The jury is still out on whether a cutoff-free QFT of the strong interactions alone is feasible, as testified by the Clay Mathematics Institute's announcement of a millennium prize for the construction of a quantum Yang–Mills theory with compact gauge group on 3 + 1 dimensional spacetime.

we should have to go to such length (in both senses of this phrase) as the Planck length $\lambda_{\text{Planck}} \approx 1.62 \times 10^{-35}$ m to truly understand a theory which was invented to understand effects that live on scales between and about, say, the electron's classical radius $\lambda_{\text{classical}} \approx 2.82 \times 10^{-15}$ m and its Compton wave length $\lambda_{\text{Compton}} \approx 3.86 \times 10^{-13}$ m, yet if there is nothing "rong" with QED other than that it ignores all non-electromagnetic effects, then this conclusion may be hard to avoid.

However, almost from its beginnings, the dissenting opinion was expressed that (nowadays standard) QED, based as it is on the quantization of the classical Maxwell–Dirac field equations,[(13, 60, 102)][7] is not properly set up, that an intrinsically well-defined quantum theory of electromagnetism is yet to be found, and that an intrinsically well-defined classical (microscopic) theory of electromagnetism may serve as important stepping stone on our way to the quantum theory. Among the founding fathers of modern quantum theory this view was held most prominently by Dirac, Schrödinger, and by Born who assessed the situation in 1933 thus (cf. Weinberg's words in the introductory quotation):

> "The attempts to combine Maxwell's equations with the quantum theory (...) have not succeeded. One can see that the failure does not lie on the side of the quantum theory, but on the side of the field equations, which do not account for the existence of a radius of the electron (or its finite energy = mass)." Max Born[(16)]

Born surmised that the infinite self-energy problems that plague QED could be eliminated by quantizing nonlinear classical electromagnetic field equations which assign point charges a finite (classical) self-energy. To find suitable field equations, he recycled some earlier ideas of Gustaf Mie who, after his better known work on the scattering of electromagnetic waves in colloidal media,[(76)] had laid down a very general Lagrangian framework for nonlinear classical electromagnetic field theories[(77)] among which Mie hoped to find one in which electrons are smooth *solitons*—a monumental series of papers which had not escaped Born's attention[(15)] (see also ref. 51, 82, and 103). Born's paper[(16)] launched an alternate quest for QED, but despite an intense pursuit, by Born and Infeld,[(16–18, 20–22)] Pryce,[(86–89)] Schrödinger,[(91–94)] and Dirac,[(37)] a disappointed Born would near the end of his life concede that "[t]he adaption of these ideas to the principles of quantum theory and the introduction of the spin has however met with no success" (ref. 19, p. 375). Ironically, only a year later the publications refs. 14 and 85 supplied a compelling new piece of evidence which suggested that Born and his prominent fellow dissenters were up to something after all, and which has suggested to us to pursue this matter a little further.

[7] Incidentally, the classical Maxwell–Dirac equations are well-behaved, see refs. 23 and 43.

To be specific, to Mie's requirements (i), that the field equations be covariant under the Poincaré group, and (ii), that they reduce to Maxwell's equations for the electromagnetic vacuum fields in the weak field limit, Born had added the postulates (iii), that the field equations be covariant under a Weyl (gauge) group, and (iv), that the electromagnetic field energy density surrounding a point charge be integrable. Postulates (i)–(iv) do not uniquely identify the field equations, but in refs. 14 and 85 it was discovered that by adding to (i)–(iv) the reasonable physical requirement (v), that the speed of light [sic!] be independent of the polarization of the wave fields,[8] one arrives at a *unique* one-parameter family of field equations, indeed the one proposed—in "one of those amusing cases of serendipity in theoretical physics" (ref. 11, p.37)—by Born and Infeld.[(20)9] In short, the Born–Infeld theory, which in essence replaces Maxwell's "pure aether"[10] by an aether with nontrivial polarizabilities to avoid the infinite electromagnetic self-contributions to the mass of a point charge, does so in a distinctly unique way.

Now, the Lorentz program of electrodynamics[(74)] fails for point charges embedded in Maxwell's pure aether because of their infinite electromagnetic self-masses and because their Lorentz self-force is "infinite in all directions."[11] Unfortunately, the Maxwell–Born–Infeld field equations lead to a Lorentz self-force on a point charge which still is undefined in magnitude and direction. Many attempts have been made[(13, 17, 21, 28, 31, 32, 37, 88, 93)] to obtain a meaningful Newtonian equation of motion for the point charges, either by *imposing* the law of energy-momentum conservation or by regularization of the electromagnetic fields, yet upon reflection (and close inspection) such attempts are found wanting. The main contribution of the present paper is a well-defined classical relativistic law of motion for spinless point charges which interact with the total classical electromagnetic Maxwell–Born–Infeld fields. This completes the consistent implementation

[8] The linear Maxwell–Lorentz equations for a point charge satisfy (i), (ii), (iii), (v), but not (iv).

[9] While the unique characterization of the Maxwell–Born–Infeld field equations in terms of (i)–(v) was apparently not known to Born and his contemporaries, the fact that these field equations satisfy, beside items (i)–(iv), also item (v) is mentioned in passing also on p.102 in ref. 92 as the absence of birefringence (double refraction).

[10] Henceforth, "aether" will be short for "electromagnetic vacuum," and we will drop the quotes.

[11] This (translated) phrase is borrowed from a passage of Emil Wiechert's monumental paper[(105)] found on pp. 41/42 in ref. 61; it also happens to be the title of the published version of Freeman Dyson's 1985 Gifford lectures in Aberdeen[(40)] (original title: "In praise of diversity"). (Both Wiechert and Dyson of course meant to indicate something more poetical.)

of the notion of the point charge in the classical relativistic theory of electromagnetism,[12] without any need for regularization or renormalization.[13]

The gist of the matter is our observation, apparently not made elsewhere before, that the classical electromagnetic potentials for the solutions of the Maxwell–Born–Infeld field equations with point sources have just enough regularity so that a relativistic Hamilton–Jacobi theory can be put to work, though not enough regularity for this Hamilton–Jacobi theory to reduce by differentiation to a (relativistic) Newtonian theory of motion driven by the total Lorentz force[14]—to which differentiation *would* reduce it *if* the electromagnetic potentials *were* differentiable, as functions of spacetime, at the world-points of the point charges. In short, within the classical domain a relativistic Hamilton–Jacobi law of motion, i.e., a first-order (velocity) differential equation for the actual positions, proves to be the more fundamental notion; the relativistic Newtonian law, i.e., a second-order (acceleration) differential equation for the positions, is a derived concept with only approximate validity.

What could seem to be merely a technical fine point that had been overlooked so far has, however, conceptual consequences: since in classical Born–Infeld electrodynamics with point charges the electromagnetic potential **A** as a function on spacetime is not differentiable at the locations of the charges, the Hamilton–Jacobi phase function[15] Φ cannot be eliminated and, hence, acquires significance in its own right. More to the point, what acquires significance is the guiding field on configuration space which in particular guides the actual configuration of the point charges, and

[12] The classical electromagnetic theory presented in this paper should not be mistaken as a merely mathematically fancier resurrection of the so-called classical electron theory of Abraham and Lorentz.(1, 74, 75, 90) In the present theory electrons are implemented as *point defects* in the electromagnetic potential field which cannot be transformed away, and which are characterized by a Poincaré- and Weyl-invariant topological quantity that is identified with the electric unit charge. Thus, in this theory the electron is a *true* point in the sense of Frenkel,(44) which remains the simplest notion compatible with the absence of empirical evidence to the contrary. In contrast, in classical electron theory the electron was assumed to have an inner structure, to unlock the secrets of which was the purpose of that theory (proven by Einstein(42) to be in vain). Yet, classical electron theory has remained an interesting dynamical theory in its own right; see refs. 2, 3, 10, 66, 67, and 96.

[13] We add that since Dirac's paper,(36) attempts to establish a classical Lorentz electrodynamics with point charges through (negative) infinite bare mass renormalization have continued. Recent rigorous contributions are refs. 5, 6, 47, and 80.

[14] Presumably this is as close as one can come to implementing the spirit of the Newtonian law of motion for point charges driven by the total Lorentz force, and yet not having quite the same.

[15] In the physics literature the Hamilton–Jacobi phase function is usually denoted by $S\ (=\hbar\Phi)$.

which is furnished by the gauge-invariant co-variant derivative[16] of Φ w.r.t. a configuration space-indexed family of **A**s. In this vein, Φ and the configuration-space indexed family of **A**s become somewhat akin to the wave function in quantum mechanics, which certainly is a surprising departure from conventional wisdom about classical electromagnetic theory.

Our theory allows us to re-assess the subtle issue of "Born's aether constant," by which name we refer to the new dimensionless physical constant that enters the Born–Infeld law of the aether. Reviving Abraham's ideas about the electromagnetic origin of the electron's inertia, a.k.a. mass, Born[(16)] proposed that the value of his aether constant be determined by identifying the empirical electron rest energy $m_e c^2$ with the now finite electrostatic energy of a point charge at rest. Born and Infeld[(20)] reinforced this thought by their erroneous result that the field energy functional is the conserved energy quantity, which is not true in the presence of point charges. But even knowledge of the correct conserved energy functional is not in itself sufficient to determine the correct value of the aether constant. Interestingly enough, the true value of the aether constant may well be known only after the theory has been successfully quantized.

This brings the discussion back to Born's original motivation, namely to develop a consistent classical electrodynamics with point charges as a stepping stone in pursuit of the elusive consistent quantum theory of electromagnetism with point charges. We here announce that we succeeded in the partial quantization of our theory (spin and photon are not yet implemented), which we take as a major encouragement to pursue the full quantization, with spin and photon, in due course. The partially quantized theory will be presented in the sequel[(63)] to this paper.

In the remainder of this paper, after we have stipulated the dimensionless units we use, we first formulate the classical theory compactly in a natural dimensionless and manifestly Poincaré- and Weyl-(gauge-)covariant manner. In particular, the basic equations of the theory are listed and explained in Section 3.2. In Section 4, the more elaborate evolutionary formalism on the space-like standard foliation is extracted from the covariant formulation, the Cauchy problem is formulated and the possibility to implement Pauli's principle discussed; also the conservation laws are stated and their proof is sketched. Then, in Section 5, for the benefit of the reader we recall the most important pertinent results about solutions with point charges that we could find in the literature; in particular, Born's field solution for a single point charge and Hoppe's field solution for an infinite crystal are listed; what seems to be known about charge-free solutions of the Maxwell–Born–Infeld field equations is collected in an appendix. There

[16] More precisely, it is the co-variant logarithmic derivative of $-ie^{i\Phi}$.

we also add the new result that genuinely electromagnetic, subluminal charge-free soliton solutions of the Maxwell–Born–Infeld field equations can at most occur if the field strengths exceed a (huge) threshold, the rigorous proof of which we outline. In Section 6 we illustrate the mathematical integrity of the Cauchy problem with point charges at hand of two examples. In Section 7 we assess Born's calculation of the value of the aether constant. The main part of the paper ends with a summary and outlook in Section 8.

2. THE DIMENSIONLESS FORMULATION

The simplest presentation is achieved if the theory is written in an economical dimensionless manner. Moreover, in order to make the eventual quantization of the classical theory that we undertake in our follow-up paper[63] as transparent as possible, we use the same units for the classical and for the quantum theory; in fact, this only helps in delineating the range of validity of the classical theory in regard to the more fundamental quantum theory. Since quantum theory is more fundamental, we use those units that quantum theory itself suggests as most natural. As a consequence of this, Sommerfeld's fine structure constant will necessarily make its appearance already in the classical theory. However, its appearance in a classical theory must not be misread as meaning that classical theory *alone* would allow us to determine the value of Sommerfeld's fine structure constant. Clearly, units involving $\hbar$ can be used only after the fact ($\hbar$), which classical physics itself knows nothing about. After these words of warning, we now proceed and list the natural units.

2.1. Natural Physical Units

In its purest, genuinely electromagnetic setting the theory deals exclusively with positively and negatively charged electrons (though spinless, for now) and electromagnetic fields in spacetime, which in turn might be flat and passive or curved and dynamical. Therefore the arguably most natural dimensionless formulation is obtained with the following conversion factors between Gaussian and dimensionless units: $\hbar$ (Planck's constant divided by 2π) for both the unit of action and the magnitude of angular momentum, e (elementary charge) for the unit of charge, m_e (electron rest mass) for the unit of mass, c (speed of light *in vacuo*) for the unit of speed. Thus, length and time are both referred to in the same units, for which the Compton wave length of the electron $\lambda_C = \hbar/m_e c$ is used to convert the dimensionless unit of length and time. Accordingly, the unit magnitude of the electromagnetic fields is to be converted by a factor e/λ_C^2, while the natural unit for the magnitude of momentum and the energy are converted, respectively, by factors $m_e c$ and $m_e c^2$.

2.2. The Universal Parameters

In the general-relativistic spacetime, the genuinely electromagnetic theory features precisely three universal dimensionless constants,[17] α, β, and γ, but only α and β figure in the special-relativistic spacetime.

The constant α is Sommerfeld's fine structure constant, i.e.,

$$\alpha \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137.036}. \tag{1}$$

Strictly speaking, since our set of dynamical equations is new, we have to vindicate the identification (1). The value of α can be determined by demanding that in the non-relativistic radiation-free limit of the classical theory the well-known law of motion for the electrons obtains (in our units). This classical dynamical vindication of (1) we will carry out non-rigorously. In our follow-up paper,[(63)] which deals with the quantum theory, we will carry out a rigorous study of the hydrogen spectrum which confirms our result (1) for α.

The parameter β is Born's aether constant,[18] which enters through the Born–Infeld aether law. Its value is a rather subtle issue. Born's result for β is[(16)]

$$\beta|_{\text{Born}} = \tfrac{1}{6}\,\mathrm{B}(\tfrac{1}{4}, \tfrac{1}{4})\,\alpha \approx 1.2361\alpha, \tag{2}$$

but as we will explain in this paper, (2) must be taken with a grain of salt. A study of the hydrogen spectrum in our follow-up paper[(63)] will supply upper bounds on β compatible with (2), but as long as spin and the photon have not been incorporated in the theory, the value for β remains tentative.

Finally, γ is the electron's "gravitational fine structure constant," given by

$$\gamma \equiv \frac{Gm_e^2}{\hbar c} \approx 1.75 \times 10^{-45} \tag{3}$$

[17] Other dimensionless parameters, like the numbers of positive and negative electrons, are better thought of as part of the initial data, which means they are non-universal parameters.

[18] Born[(16)] originally introduced the dimensional parameter a, but subsequently switched to use $b \equiv a^{-1}$. Our dimensionless $\beta^2 \propto a$ (the reason for the "square" is simply to avoid having to write some awkward "square roots" later on).

and obtained by rewriting Einstein's field equations in our dimensionless units; here, G is Newton's constant of universal gravitation. We will focus mostly on the genuinely electromagnetic theory in the special-relativistic limit $\gamma \downarrow 0$, choosing Minkowski spacetime as solution for the vacuum Einstein equations.

Some non-genuine yet electromagnetically important phenomena, such as nuclei charged with z electric units and having considerably bigger masses than electrons due to their strong interactions, can be readily accommodated[19] at the price of introducing additional dimensionless (effective) parameters into the theory, viz. the very z's and the relative masses of the nuclei. We will outline the modifications in a separate section. These modifications should be derivable from some deeper theory with fewer parameters which ultimately would be truly universal.

3. THE COVARIANT FORMALISM

3.1. The Minkowski Spacetime

3.1.1. The Manifold and Its Tangent Space

Except in a brief section, in this paper the arena for the electromagnetic phenomena is the passive Minkowski spacetime $\mathbb{M}^4$, a pseudo-Riemannian flat, oriented 4-manifold with Lorentzian metric tensor $\mathbf{g}$ of signature[20] $+2$. The points ϖ in Minkowski spacetime are called world-points (or sometimes, events).

The geometry of Minkowski spacetime is defined through the Poincaré group acting on $\mathbb{M}^4$, consisting of the isometries with respect to $\mathbf{g}$, i.e., the spacetime translations, rotations, and reflections.[8, 97] Hence, we can choose any particular world-point $\varpi_0 \in \mathbb{M}^4$ and identify any other world-point $\varpi \in \mathbb{M}^4$ with the world-vector $\mathbf{w} \in \mathbb{T}_{\varpi_0}(\mathbb{M}^4)$ by stipulating that $\mathbf{w}$ is the oriented chord from ϖ_0 to ϖ. In this way $\mathbb{M}^4$ becomes identified with $\mathbb{T}_{\varpi_0}(\mathbb{M}^4)$, its tangent space at ϖ_0.

A Lorentz frame is a basis for $\mathbb{T}_{\varpi_0}(\mathbb{M}^4)$, viz. any tetrad $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of fixed unit world-vectors in $\mathbb{T}_{\varpi}(\mathbb{M}^4)$ that satisfy the elementary inner product rules

$$\mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \equiv \mathbf{e}_\mu \cdot \mathbf{e}_\nu = \begin{cases} -1 & \text{for} \quad \mu = \nu = 0 \\ 1 & \text{for} \quad \mu = \nu > 0 \\ 0 & \text{for} \quad \mu \neq \nu. \end{cases} \tag{4}$$

[19] Particles which do not interact at all electromagnetically can be readily accommodated, too.

[20] Here we follow mathematical conventions as in ref. 49; in the physics literature this corresponds to saying that the signature of $\mathbf{g}$ is $(-, +, +, +)$.

With respect to such a Lorentz frame any real-valued vector $\mathbf{w} \in \mathbb{T}_{\varpi_0}(\mathbb{M}^4)$ has the representation (Einstein summation convention understood)

$$\mathbf{w} = w^\mu \mathbf{e}_\mu \tag{5}$$

with $w^0 = -\mathbf{w}\cdot\mathbf{e}_0$ and $w^\mu = \mathbf{w}\cdot\mathbf{e}_\mu$ for $\mu = 1, 2, 3$. The identification $\mathbf{w} \cong (w^0, w^1, w^2, w^3) \in \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the set of ordered real 4-tuples equipped with Minkowski metric, provides a global coordinatization for $\mathbb{M}^4$.

World-vectors $\mathbf{w}$ are classified into space-like, light-like, and time-like according as $\mathbf{w}^{\cdot 2} > 0$, $\mathbf{w}^{\cdot 2} = 0$, or $\mathbf{w}^{\cdot 2} < 0$, where $\mathbf{w}^{\cdot 2} \equiv \mathbf{w}\cdot\mathbf{w}$. A space-like world-vector is connected via a continuous orbit of the Lorentz group with a non-zero world-vector $\mathbf{s}$ satisfying $\mathbf{s}\cdot\mathbf{e}_0 = 0$, a time-like one with a non-zero world-vector $\mathbf{t}$ satisfying $\mathbf{t}\cdot\mathbf{e}_\mu = 0$ for all $\mu = 1, 2, 3$. The light-like world-vectors form the light-(double)cone in $\mathbb{T}_{\varpi_0}(\mathbb{M}^4)$, which separates the space-like from the time-like world-vectors in $\mathbb{T}_{\varpi_0}(\mathbb{M}^4)$.

3.1.2. World-Lines and Point-Histories in Spacetime

One-dimensional geometrical objects in the Minkowski spacetime, viz. continuous curves in spacetime, are called *world-lines*. A time-like future-oriented world-line is called a *point-history*, or just history for short, and denoted H.

We shall assume that histories are of class C^1, in fact $C^{1,\alpha}$. In that case, at each event on a history there is a unique future-oriented unit tangent vector to the history. Its metric dual is a co-vector $\mathbf{u}$, satisfying $\mathbf{u}\cdot\mathbf{u} = -1$ and $u_0 > 0$.

By $\varpi_p \mathrm{H} \varpi_f$ we denote a truncated point history between two events, ϖ_p and ϖ_f, where ϖ_f lies in the future of ϖ_p. By integrating the co-vector $\mathbf{u}$ along a truncated history, we obtain the invariant *proper-time span* $\mathscr{T}[\varpi_p \mathrm{H} \varpi_f]$ of the truncated history, $\mathscr{T}[\varpi_p \mathrm{H} \varpi_f] = \int_{\varpi_p \mathrm{H} \varpi_f} \mathbf{u}$. In particular, by choosing any convenient event ϖ_0 on a history as reference world-point to which is assigned the proper-time Null, we can then assign to each event ϖ on that history a proper-time $\tau \in \mathbb{R}$ relative to ϖ_0 defined to be the proper-time span $\mathscr{T}[\varpi_0 \mathrm{H} \varpi]$ $(-\mathscr{T}[\varpi \mathrm{H} \varpi_0)])$ if ϖ lies in the future (past) of ϖ_0. Since the history is time-like future-oriented, its proper-time assignment is one-to-one onto and order-preserving. Hence, proper-time in turn can serve as a natural parameterization for the history, i.e., a history is given by a $C^{1,\alpha}$ mapping $\eta\colon \tau \mapsto \varpi = \eta(\tau) \in \mathbb{M}^4$. Similarly, the metric dual of the unit tangent vector at each world-point ϖ on the history is then a co-vector-valued C^α map $\tau \mapsto \mathbf{u}(\tau)$.

3.1.3. Space-Like Slices

Dual as a concept to point-histories in spacetime are space-like slices of spacetime. A space-like slice $\Sigma \subset \mathbb{M}^4$ is a three-dimensional simply connected hypersurface in $\mathbb{M}^4$ with a time-like normal vector at every point in Σ. Topologically, Σ is homeomorphic to $\mathbb{R}^3$. Without loss of generality one can assume that $\Sigma_t = \{\varpi: T(\varpi) = t\}$ is the boundary of a level set of a differentiable function $T: \mathbb{M}^4 \to \mathbb{R}$ which has the properties that $\mathrm{ran}(T) = \mathbb{R}$ and that the metric dual to $\mathbf{d}T(\varpi)$ is time-like for all ϖ in $\mathbb{M}^4$; here $\mathbf{d}$ denotes E. Cartan's exterior derivative on $\mathbb{M}^4$.

3.1.4. Space-Like Foliations

The light-cone structure of $\mathbb{M}^4$ allows one to foliate this manifold with space-like leafs. The intersection of any two such space-like hypersurfaces Σ_{t_1} and Σ_{t_2} with $t_1 \neq t_2$ is empty, and each ϖ is in precisely one Σ_t for some suitable $t \in \mathrm{ran}(T)$. As a consequence, $\bigcup_{t \in \mathbb{R}} \Sigma_t$ can be identified with all of spacetime, and this identification constitutes the foliation of spacetime (generated by T).

We can coordinatize all leafs Σ_t, $t \neq 0$, with the coordinates on Σ_0 by following the integral curves of the metric dual of $\mathbf{d}T$ to their piercing through points in Σ_0. Equipped with such a global parameterization of a foliation, Minkowski spacetime is always diffeomorphic to the product manifold $\mathbb{R} \times \Sigma_0$. If (s^1, s^2, s^3) are arbitrary coordinates on the space-like Σ_0, we can coordinatize all of $\mathbb{M}^4$ by $\varpi \cong (s^0, s^1, s^2, s^3)$, with $s^0 = t$. The spacetime metric of $\mathbb{M}^4$ with respect to the foliation generated by T in these coordinates is given by the world-line element

$$\mathrm{d}s^2 = -\ell^2(\varpi)\,\mathrm{d}t^2 + \sum_{1 \leqslant m,n \leqslant 3} g_{mn}(\varpi)\,\mathrm{d}s^m\,\mathrm{d}s^n, \tag{6}$$

where $\ell = (-\mathbf{d}T \cdot \mathbf{d}T)^{-1/2}$ is the *lapse function*, and where the $g_{mn}(\varpi)$ are the components, with respect to an arbitrary space frame of Σ_t, of the *first fundamental form* g of the leaf at ϖ, which is induced on Σ_t by the Minkowski metric on $\mathbb{M}^4$.

3.2. The Electromagnetic Spacetime

We are now ready to define classical *electromagnetic (flat) spacetime* as Minkowski spacetime equipped with a classical electromagnetic structure, consisting of a two-form (field tensor) which is defined almost everywhere, the exception being N time-like oriented complete world-lines (the histories of the point charges). Any such electromagnetic structure satisfies the Maxwell–Born–Infeld laws of the electromagnetic field with point sources, which we first state and then assess.

3.2.1. The Maxwell–Born–Infeld field Laws with Point Sources

Let $\bigcup_k \mathrm{H}_k$ denote the set of N point histories with which $\mathbb{M}^4$ is threaded. Then on $\mathbb{M}^4 \backslash \bigcup_k \mathrm{H}_k$ there is stipulated to exist a two-form $\mathbf{F}$, called Faraday's electromagnetic field tensor. The postulate that $\mathbf{F}$ is closed on $\mathbb{M}^4$, i.e.,

$$\mathbf{dF} = \mathbf{0} \tag{7}$$

in the sense of distributions, is known as the *Faraday–Maxwell law* (cf. ref. 78).

To relate the two-form $\mathbf{F}$ to the point histories, $\mathbf{F}$ is mapped to another two-form $\mathbf{M}$, called Maxwell's electromagnetic displacement tensor. The map is effected by the *Born and Infeld law of the aether*,

$$-{}^{\star}\mathbf{M} = \frac{\mathbf{F} - \beta^{4\star}(\mathbf{F}\wedge\mathbf{F})\ {}^{\star}\mathbf{F}}{\sqrt{1-\beta^{4\star}(\mathbf{F}\wedge{}^{\star}\mathbf{F}) - \beta^{8}({}^{\star}(\mathbf{F}\wedge\mathbf{F}))^{2}}}, \tag{8}$$

where ${}^{\star}\mathbf{F}$ (etc.) is the Hodge dual of $\mathbf{F}$ (etc.). The rather complicated law (8), reduces to the linear law $\mathbf{M} = {}^{\star}\mathbf{F}$ of Maxwell's pure aether in the limit[21] $\beta \downarrow 0$.

The distributional exterior derivative of the two-form $\mathbf{M}$ now gives the electromagnetic current density,

$$\mathbf{dM} = 4\pi \mathbf{J}. \tag{9}$$

Equation (9) will be called the *Ampére–Coulomb–Maxwell* law; cf. ref. 78.

Equation (9) implies that the electromagnetic current density $\mathbf{J}$ is a closed three-form, i.e., its exterior derivative on $\mathbb{M}^4$, in the sense of distributions, vanishes,

$$\mathbf{dJ} = \mathbf{0}. \tag{10}$$

Equation (10) is known as the *law of the conservation of electric charge*.

In the genuinely electromagnetic setting of the theory, all charges are either positive or negative unit point charges, representing electrons of

[21] More precisely, since β is a universal physical constant which has to be assigned a nonzero value to avoid the infinite self-energies of the classical Lorentz electrodynamics, the proper way of putting it is to say that (8) reduces to Maxwell's linear law of the aether whenever the magnitude of $\mathbf{F}$ is sufficiently small; i.e., when $\beta^4\,|{}^{\star}(\mathbf{F}\wedge{}^{\star}\mathbf{F})| + \beta^8({}^{\star}(\mathbf{F}\wedge\mathbf{F}))^2 \ll 1$, then $\mathbf{M} \sim {}^{\star}\mathbf{F}$ (weak field limit).

either variety. For a system of $N \geqslant 0$ electric unit point charges the electromagnetic current density $\mathbf{J}(\varpi)$ at ϖ is given by the familiar expression,[78, 99]

$$\mathbf{J}(\varpi) = \sum_{k \in \mathcal{N}} \int_{-\infty}^{+\infty} z_k \, {}^\star\mathbf{u}_k(\tau) \, \delta_{\eta_k(\tau)}(\varpi) \, \mathrm{d}\tau, \tag{11}$$

cf. also refs. 87 and 101. Here, $\delta_{\eta(\tau)}(.)$ is the Dirac measure on $\mathbb{M}^4$ concentrated at $\eta(\tau)$ at proper-time τ, ${}^\star\mathbf{u}_k(\tau)$ is the Hodge dual of the future-oriented Minkowski-velocity co-vector $\mathbf{u}_k(\tau)$, and z_k is the sign of the kth point charge;[22] furthermore, $\mathcal{N} \subset \mathbb{N} \cup \{0\}$ is the set of N indices. We set $\mathcal{N} \equiv \varnothing$ if $N = 0$, in which case $\sum_{k \in \varnothing}(...) \equiv 0$, so that the charge-free situation is included in (11).

The manifestly covariant character of the Maxwell–Born–Infeld laws makes it plain that, whichever particular electromagnetic structure satisfying these laws may be the actually realized one in Minkowski spacetime, any two Lorentz observers of this electromagnetic spacetime would necessarily conclude that they see their respective Lorentz frame manifestations of the *same* electromagnetic spacetime, whatever their relative state of uniform motion with respect to each other might be.

3.2.2. Limitations of the Maxwell–Born–Infeld Field Laws with Point Sources

For any particular electromagnetic spacetime as defined above, a-priori knowledge of F would in principle allow us to compute from it the remaining electromagnetic quantities M and $\mathbf{J}$, the latter giving $\bigcup_k \mathrm{H}_k$ and the $\mathbf{u}_k(\tau)$. Of course, we have no complete a-priori knowledge of the "actual" F (i.e., "actual" to the extent that curvature and non-electromagnetic effects can be neglected), but the partial knowledge we do have of our local electromagnetic spacetime indicates much more order in the world than demanded by the Maxwell–Born–Infeld field laws alone. In particular, even granted we would have complete knowledge of the past and present part of the electromagnetic spacetime with respect to some space-like hypersurface of some standard Lorentz frame (and also granted we would know F at space-like infinity), the Maxwell–Born–Infeld laws would not allow us to continue that information uniquely into the future to extend the knowledge of the electromagnetic spacetime, not to mention completing the

[22] This formula for $\mathbf{J}$ reveals very nicely that the "sign of the point charge" z_k is actually an artifact of our insistence (psychologically inherited from how we perceive the world around and in us) that the $\mathbf{u}_k$ be future oriented. Geometrically it is much more natural to absorb z_k into $\mathbf{u}_k$ and simply distinguish future- and past-oriented $\mathbf{u}_k$s, as Wheeler has suggested.

knowledge; unless, that is, **J** would vanish identically. But since a basic hypothesis of Born–Infeld's, and also our, theory is that there are point charges, in order to be able to uniquely continue such data of the electromagnetic spacetime we need to supply a mathematically well-defined law for the point histories, and not just any such law but one which accounts for the observed regularities of point charge motions. In particular, the law for the point histories has to reduce to the known asymptotic law in the limit of gently accelerated, slowly moving, far separated point charges, which is Newton's law of motion equipped with the Coulomb law for the forces between the point charges; we also know that magnetic effects can be taken into account very accurately (in the same regime of motion) by replacing the Coulomb force on a particle with the Lorentz force due to all other particles.

Unfortunately, the formal expression of Newton's law of motion equipped with the total Lorentz force involves **F** at the location of the point charges, but **F** inherits from **M** (recall (9)) those locations as singular points; and while in contrast to the singularities of **M** those of **F** are very mild (finite discontinuities) when $\beta \neq 0$, the tensor field **F** cannot be extended continuously into those points. This renders the Newtonian law of motion with the total electromagnetic fields an only formal arrangement of symbols without proper meaning. We need a crucial new insight.

3.3. A First-Order Law for the Histories of the Point-Charges

To guide our search for the law of motion, we argue heuristically as follows. First of all, we recall that (7) implies that, except at the locations of the point charges where it is necessarily undefined, an otherwise differentiable field tensor **F** is an exact two-form on $\mathbb{M}^4 \setminus \bigcup_k \mathrm{H}_k$, i.e.,

$$\mathbf{F} = \mathbf{dA}, \tag{12}$$

where $\mathbf{A}$, the *electromagnetic potential*, is a one-form on $\mathbb{M}^4 \setminus \bigcup_k \mathrm{H}_k$. Recall also that such an $\mathbf{A}$ is not uniquely defined by (12), for adding an exact one-form to $\mathbf{A}$, i.e.,

$$\mathbf{A} \to \mathbf{A} + \mathbf{d}\Upsilon, \tag{13}$$

where $\Upsilon \colon \mathbb{M}^4 \to \mathbb{R}$ is any zero-form, obviously leaves **F** unchanged. The interesting point now is that, since the singularities of such an **F** are mild, namely finite jump discontinuities, we can assume that the one-form $\mathbf{A}$ can be Lipschitz-continuously extended into the locations of the point charges, hence onto all of $\mathbb{M}^4$, and (12) is then to be understood in the sense of distributions. Actually, what we have just been saying is rigorously known to be true only for a point charge in arbitrary uniform motion, but it is very much expected to be true also for accelerated point charges, for in the immediate vicinity of the accelerated point charge its field should look like the co-moving stationary field of a charge in uniform motion with same instantaneous velocity, while the acceleration makes only a *relatively small* correction to the fields. Although the intuition behind this expectation is borrowed from the experience with the Liénard–Wiechert solutions[(72, 106)] to the linear Maxwell–Lorentz equations with point charge source, *in this respect* the nonlinear Maxwell–Born–Infeld field equations should behave just like the linear Maxwell–Lorentz equations;[23] unless, that is, shocks would form. Fortunately, shock formation is ruled out by the complete linear degeneracy of the linearized Maxwell–Born–Infeld field equations.[(14, 85)] It follows that, if all what we just said is indeed rigorously true for any electromagnetic spacetime satisfying the Maxwell–Born–Infeld field laws with point charges, then the familiar canonical momentum of the kth point charge,

$$\mathbf{P}_k(\tau) = \mathbf{u}_k(\tau) + z_k \alpha \mathbf{A}(\eta(\tau)), \tag{14}$$

is well-defined along its point history H_k in any such an electromagnetic spacetime. In any event, (14) is certainly well-defined for those parts of an electromagnetic spacetime satisfying the Maxwell–Born–Infeld field laws with point charges for which what we said is true. After having stressed this, we will now take (14) as our point of departure for constructing a well-defined *first-order* law for the point histories.

To prepare the ground for the statement of the law of motion, at this point it is helpful to recall the co-variant formulation of the *test particles* Hamilton–Jacobi theory of Newton's law of motion for point charges in *given* nice (read: at least Lipschitz continuous) electromagnetic fields on $\mathbb{M}^4$ so that the electromagnetic potentials are differentiable. Next we upgrade this test particles Hamilton–Jacobi formalism to nice electromagnetic fields which are to be *solved for alongside* the Hamilton–Jacobi equation. While this formulation is inadequate for our situation in which the electromagnetic fields are not Lipschitz continuous at the locations of the actual point charges, a brief analysis of why the "upgraded test particle Hamilton–Jacobi formulation" fails will right away lead us to the remedy, which we call the *proper particles* Hamilton–Jacobi formulation.

[23] Clearly, a rigorous proof of this assertion needs to be supplied at some point.

3.3.1. Test Particles Hamilton–Jacobi Theory

In this vein, we temporarily pretend that $\mathbf{A}$ in (14) were not the electromagnetic one-form for the total electromagnetic Maxwell–Born–Infeld fields created by the point charges, but instead just a smoothly differentiable electromagnetic potential field on $\mathbb{M}^4$ for some given, nice electromagnetic fields. Then the familiar[24] test particles Hamilton–Jacobi theory sets $\mathbf{P}_k(\tau) = (\mathbf{d}_k\tilde{\Phi})(\eta_1(\tau),..., \eta_N(\tau))$ for $\tilde{\Phi}(\varpi_1,..., \varpi_N)$ a scalar field on world configuration space $\mathbb{M}^{4N}$, so that (14) becomes

$$\mathbf{u}_k(\tau) = \mathbf{d}_k\tilde{\Phi}(\varpi_1,..., \varpi_N) - z_k\alpha\mathbf{A}(\varpi_k)|_{\{\varpi_n = \eta_n(\tau)\}}, \tag{15}$$

which needs to be solved for all k, with initial data for the $\eta_k(0)$ supplied. Interestingly, the scalar field $\tilde{\Phi}$ in turn is implicitly determined by (15), too, for notice that the r.h.s. of (15) has to produce a future-oriented co-vector $\mathbf{u}_k$ satisfying $^\star(\mathbf{u}_k \wedge {}^\star\mathbf{u}_k) = 1$. But since the world-line for $\mathbf{u}_k$ is not known *a priori*, the r.h.s. of (15) can only do what it is supposed to do if $\tilde{\Phi}$ on $\mathbb{M}^{4N}$ satisfies the N equations

$$^\star((\mathbf{d}_k\tilde{\Phi}(\varpi_1,..., \varpi_N) - z_k\alpha\mathbf{A}_k(\varpi_k)) \wedge {}^\star(\mathbf{d}_k\tilde{\Phi}(\varpi_1,..., \varpi_N) - z_k\alpha\mathbf{A}(\varpi_k))) = 1. \tag{16}$$

Each Eq. (16) has a double root, of which the future-oriented one is to be chosen. This system of equations is consistent if the N world-points ϖ_k are distributed over a space-like hypersurface in $\mathbb{M}^4$. However, since in given fields each test particle moves as if it were the only one around, we may separate variables as $\tilde{\Phi}(\varpi_1,..., \varpi_N) = \sum_{k=1}^N \tilde{\Phi}^{(k)}(\varpi_k)$, where $\tilde{\Phi}^{(k)}(\varpi_k)$ is a genuine function on the kth component of $\mathbb{M}^{4N}$. In that case $\mathbf{d}_k\tilde{\Phi}(\varpi_1,..., \varpi_k,..., \varpi_N) = \mathbf{d}_k\tilde{\Phi}^{(k)}(\varpi_k)$, and the N Eqs. (15) become individual equations for the $\tilde{\Phi}^{(k)}$, and no compatibility issue arises. By differentiating (15) with respect to (proper) time and using (16) and (12), we recover Newton's law of motion for the "actual test particles" which are accelerated by the Lorentz force of the given electromagnetic fields.

Notice that the test particles Hamilton–Jacobi equation (16) in given fields does not depend on the motion of the actual test point charges. Equation (16) is solved autonomously with initial data for $\tilde{\Phi}$ compatible

[24] Perhaps it will become explicitly obvious only in the space and time decomposition, and after synchronization w.r.t. Lorentz time, that these co-variant equations describe the familiar N test particles Hamilton–Jacobi formalism. The manifest simplicity of the co-variant formalism should leave no doubt, however, that this is the natural language for relativistic Hamilton–Jacobi theory.

with the initial velocities of the N actual test particles at their starting positions. Subsequently the guiding Eq. (15) is solved to obtain the actual motions.

3.3.2. Upgraded Test Particles Hamilton–Jacobi Theory

Less well-known is the fact that, if the actual electromagnetic fields are not *a priori* given but nevertheless remain nice at the locations of the point charges, then the test particles Hamilton–Jacobi formalism can be adapted to the computation of the motion of the point charges alongside the unknown fields which they generate. We stress that this excludes fields satisfying the Maxwell–Lorentz and Maxwell–Born–Infeld field equations with point charge sources, but there are plenty of other constitutive relations between $\mathbf{F}$ and $\mathbf{M}$ which, when used in place of the Born–Infeld aether laws (8), ensure that $\mathbf{F}$ is at least Lipschitz continuous at the locations of the point charges. For instance, Infeld[55, 56] has proposed nonlinear aether laws that would seem to qualify in this respect (however, the (say) Maxwell–Infeld field equations produce birefringence and other unwanted phenomena which seem to disqualify them in these respects). The main difference, as compared to the previous case of test particle motion in given fields, is that now the Eqs. (15) and (16) are to be solved simultaneously, i.e., now the Hamilton–Jacobi partial differential Eq. (16) does depend on the actual motion. At the end of the day, almost all streamlines of the velocity field in generic configuration space represent motions of test particles in the actual field; however, precisely one such streamline coincides with the path of the actual configuration that provides the sources for the actual field, so that this particular "test particles" motion is not a test particles motion at all. To distinguish this situation from the one with given fields it may be appropriate, perhaps, to speak now of an "upgraded test particles Hamilton–Jacobi theory."

We remark that as in the test particles Hamilton–Jacobi theory, also here we may separate variables through $\tilde{\Phi}(\varpi_1,..., \varpi_N) = \sum_{k=1}^{N} \tilde{\Phi}^{(k)}(\varpi_k)$. Again the N Eqs. (15) become individual equations for the $\tilde{\Phi}^{(k)}$, but now they are still coupled through the unknown electromagnetic potential which the N actual motions generate. That this is general enough to obtain all the actual motions from upgraded test particles Hamilton–Jacobi theory follows again from the simple fact that differentiating (15) with respect to proper time, and using (16) and (12), yields Newton's law for the actual motions of the particles, but now accelerated by the Lorentz force in their self-generated electromagnetic fields, to be computed alongside the actual motions.

3.3.3. Limitations of the Upgraded Test Particles Hamilton–Jacobi Theory

Unfortunately, the upgraded test particles Hamilton–Jacobi formalism fails to give a well-defined law of motion when coupled to the electromagnetic potentials for the total electromagnetic fields that satisfy the Maxwell–Born–Infeld field equations with point sources. The reason is simply that at the end of the day only one of the test particles' trajectories in configuration space can coincide with the actual particles' trajectory, while the remaining trajectories—and that means almost all—indeed represent just motions of test particles which have no influence on the field that determines their motion. But this means that almost all these test particles configurations do not sample the actual $\mathbf{A}$ (it is understood that the "actual" $\mathbf{A}$ requires the stipulation of a gauge) the way the configuration of the N actual particles does, the exception occurring when the test particles world-configuration coincides with the actual one. As a consequence of this, the actual $\mathbf{A}$ in (16) which acts as a field on the kth copy of $\mathbb{M}^4$ is exactly as non-differentiable at the actual particles' world-points as the actual $\mathbf{A}$ is as a field on Minkowski spacetime itself. Now $\tilde{\Phi}$ in (16) inherits this non-differentiability from the actual $\mathbf{A}$, and that simply means that (15) ceases to be well-defined precisely at the actual positions where it is needed.

Fortunately, the above analysis of why the upgraded test particles Hamilton–Jacobi formalism fails with the electromagnetic potentials of the total Maxwell–Born–Infeld fields with point sources hints clearly at what is needed. Namely, instead of using a formalism in which almost all the generic configurations represent virtual test particles configurations which are not the sources for the field they sample, and which is the actual field, we should be working with a formalism in which each generic configuration represents the point sources for the very field that it samples, and which therefore will generally not be the actual one. We will call this the *proper many-body* Hamilton–Jacobi theory. Curiously, the pertinent Hamilton–Jacobi literature is silent about such a formalism,[25] even though—with hindsight—it could have been developed long ago!

3.3.4. Proper Many-Particles Hamilton–Jacobi Theory

With the help of the geometrical language of forms, a relativistically covariant proper many-body Hamilton–Jacobi theory of motion can be set

[25] Barut[(8)] seems quite pessimistic about the prospects of achieving a relativistic proper many-body Hamilton–Jacobi theory and proposes as alternative the Fokker–Schwarzschild–Tetrode-/-Feynman–Wheeler electrodynamics;[(104)] ch. VI in ref. 8. That theory does not pose an initial value problem and, hence, departs ultra-radically from conventional modes of thought about physics; for rigorous results on two-particle scattering, see ref. 9.

up as follows. Once again, that a Hamilton–Jacobi structure is behind this formulation may not be any more obvious than for the (plain or upgraded) test particles versions, but it will be unveiled in our sections on the evolutionary formalism in space and time decomposition.

According to what we wrote in our last section, the proper many-particles Hamilton–Jacobi formalism does not work with the a-priori unknown actual fields on $\mathbb{M}^4$ but with an (also a-priori unknown) family of fields on $\mathbb{M}^4$ which is indexed by N world-points; hence instead of the actual $\mathbf{J}$, $\mathbf{M}$ (whence, $\mathbf{F}$) and $\mathbf{A}$ on $\mathbb{M}^4$ we consider fields ${}^\sharp\mathbf{J}$, ${}^\sharp\mathbf{M}$, ${}^\sharp\mathbf{F}$, and ${}^\sharp\mathbf{A}$ with arguments $\in \mathbb{M}^4 \times \mathbb{M}^{4N}_{\neq}$, where $\mathbb{M}^{4N}_{\neq}$ is the configuration space of N ordered world-points with co-incidence points removed. These ${}^\sharp$fields are of course to be associated with the actual fields in a natural way, and the obvious requirement which seems to suggest itself is to demand that, when the generic configuration point $(\varpi_1,\ldots,\varpi_N) \in \mathbb{M}^{4N}_{\neq}$ in the argument of a ${}^\sharp$field is replaced by an actual configuration point $(\eta_1(\tau),\ldots,\eta_N(\tau)) \in \mathbb{M}^{4N}_{\neq}$, then this so "conditioned" ${}^\sharp$field becomes the corresponding actual field on $\mathbb{M}^4$ in the remaining world-point variable $\varpi \in \mathbb{M}^4$, i.e., ${}^\sharp\mathbf{A}(\varpi, \eta_1(\tau),\ldots,\eta_N(\tau)) = \mathbf{A}(\varpi)$, etc. However, to ask for this much is actually asking just a little too much, for by re-parametrization of the point histories we can arrange that $(\eta_1(\tau),\ldots,\eta_N(\tau))$ is any configuration point that can be formed by picking one arbitrary point from each of the N actual point histories without changing $\mathbf{A}(\varpi)$ (etc.). This in turn would imply that the individual directional derivatives of ${}^\sharp\mathbf{A}(\varpi, \varpi_1,\ldots,\varpi_N)$ along each of the N actual point histories would have to vanish. But all we really need to ask for is that for any $N+1$ world-points $(\varpi, \varpi_1,\ldots,\varpi_N)$ picked from any leaf of a foliation we have ${}^\sharp\mathbf{A}(\varpi, \eta_1(\tau),\ldots,\eta_N(\tau)) = \mathbf{A}(\varpi)$, etc. This also fixes in a canonical way the field equations for the ${}^\sharp$fields on $\mathbb{M}^4 \times \mathbb{M}^{4N}_{\neq}$ synchronized w.r.t. the foliation. Since we need a foliation for this, we postpone the presentation of these ${}^\sharp$field equations until we work out the formalism in standard foliation.

Next we form N fields $\tilde{\mathbf{A}}_k$ on synchronized $\mathbb{M}^{4N}_{\neq}$, this time by conditioning ${}^\sharp\mathbf{A}$ by successively replacing the world-point ϖ in its first argument with the N components ϖ_k of the generic configuration point; more precisely, we define $\tilde{\mathbf{A}}_k(\varpi_1,\ldots,\varpi_N) \equiv {}^\sharp\mathbf{A}(\varpi_k, \varpi_1,\ldots,\varpi_N)$ for each $k = 1,\ldots,N$. Since we may think of ${}^\sharp\mathbf{A}(\varpi, \varpi_1,\ldots,\varpi_N)$ essentially as the electromagnetic potential field for N point sources at $(\varpi_1,\ldots,\varpi_N)$, the field $\tilde{\mathbf{A}}_k(\varpi_1,\ldots,\varpi_N)$ samples ${}^\sharp\mathbf{A}(\varpi, \varpi_1,\ldots,\varpi_N)$ exactly at its kth source point.

We now formulate a law of motion for the actual configuration point in synchronized $\mathbb{M}^{4N}_{\neq}$ of the N point charges in $\mathbb{M}^4$ which is similar in appearance to (15). Namely, we replace $\mathbf{A}(\varpi_k)$ in (15) by $\tilde{\mathbf{A}}_k(\varpi_1,\ldots,\varpi_N)$ and obtain the system of first order guiding laws

$$\mathbf{u}_k(\tau) = \mathbf{d}_k\tilde{\Phi}(\varpi_1,\ldots,\varpi_N) - z_k\alpha\tilde{\mathbf{A}}_k(\varpi_1,\ldots,\varpi_N)\big|_{\{\varpi_n = \eta_n(\tau)\}} \tag{17}$$

for the actual world-points $\varpi_k = \eta_k(\tau)$ on the point-histories H_k, with $k = 1,\dots, N$. Since (17) must produce a future-oriented co-vector $\mathbf{u}_k$ satisfying $^\star(\mathbf{u}_k \wedge {}^\star\mathbf{u}_k) = 1$, the phase function $\tilde{\Phi}$ on $\mathbb{M}^{4N}_{\neq}$ must satisfy the N equations

$$^\star((\mathbf{d}_k\tilde{\Phi} - z_k\alpha\tilde{\mathbf{A}}_k) \wedge {}^\star(\mathbf{d}_k\tilde{\Phi} - z_k\alpha\tilde{\mathbf{A}}_k)) = 1, \tag{18}$$

of each of which the future-oriented root is to be chosen. These N equations are to be understood w.r.t. the same foliation as the $^\sharp$fields, for otherwise the N equations would in general over-determine $\tilde{\Phi}$ when $N > 1$. While (17) and (18) are manifestly Poincaré-invariant, the foliation is not but takes a status akin to the world lines. We add that (17) and (18) are also manifestly gauge (Weyl-)invariant. Namely, since $^\sharp\mathbf{A}$ inherits its gauge transformation from (13), a gauge transformation becomes

$$\tilde{\mathbf{A}}_k(\varpi_1,\dots, \varpi_N) \to \tilde{\mathbf{A}}_k(\varpi_1,\dots, \varpi_N) + (\mathbf{d}\Upsilon)(\varpi_k) \tag{19}$$

$$\tilde{\Phi}(\varpi_1,\dots, \varpi_N) \to \tilde{\Phi}(\varpi_1,\dots, \varpi_N) + \sum_k z_k\alpha\Upsilon(\varpi_k), \tag{20}$$

and this obviously leaves $\mathbf{F}$, $^\sharp\mathbf{F}$, and the $\mathbf{u}_k$ invariant.

We remark that while in general $\tilde{\mathbf{A}}_k(\varpi_1,\dots, \varpi_N)$ is a genuine function of all ϖ_ℓ, $\ell = 1,\dots, N$, in a wide variety of "decoherent" situations we may assume that $\tilde{\mathbf{A}}_k$ depends only on ϖ_k, and in that case we can separate variables and demand that $\tilde{\Phi}(\varpi_1,\dots, \varpi_N) = \sum_{k=1}^N \tilde{\Phi}^{(k)}(\varpi_k)$, where $\tilde{\Phi}^{(k)}(\varpi_k)$ is a genuine function on the kth component of $\mathbb{M}^{4N}_{\neq}$, i.e., $\tilde{\Phi}^{(k)}(\varpi_k)$ is not indexed by the $N-1$ other component world-points. In that case we have $\mathbf{d}_k\tilde{\Phi}(\varpi_1,\dots, \varpi_k,\dots, \varpi_N) = \mathbf{d}_k\tilde{\Phi}^{(k)}(\varpi_k)$, and the N Eqs. (17) become individual equations for the $\tilde{\Phi}^{(k)}$, though still coupled through the family of electromagnetic potentials. This decoherent situation is closely reminiscent of the upgraded test particles formulation, and yet different.

To summarize, proper many-body Hamilton–Jacobi theory is formulated with the help of a field $^\sharp\mathbf{A}$ on synchronized $\mathbb{M}^4 \times \mathbb{M}^{4N}_{\neq}$. The equation for $^\sharp\mathbf{A}$ is induced by the requirement that insertion of the actual configuration in place of the generic one produces the actual $\mathbf{A}$ for the actual Maxwell–Born–Infeld fields, which concept is well-defined w.r.t. some foliation. The field $^\sharp\mathbf{A}$ induces N fields $\tilde{\mathbf{A}}_k$ on synchronized $\mathbb{M}^{4N}_{\neq}$, with the help of which an N Minkowski-velocities field on synchronized $\mathbb{M}^{4N}_{\neq}$ can be defined, the kth component of which is given by the r.h.s. of (17), at generic world configuration points. The requirement that the components are Minkowski-velocity co-vectors gives the N Eqs. (18), defined w.r.t. the same foliation with respect to which the equation for $^\sharp\mathbf{A}$ is formulated. The actual dynamics of the N point charges is formulated as the motion of a single point in synchronized $\mathbb{M}^{4N}_{\neq}$ which moves according to a system of covariant guiding laws, the kth component of which is (17).

We end by emphasizing one more time that the fields $\tilde{\Phi}$ and $^{\sharp}\mathbf{A}$, understood with respect to a foliation, are elevated from the status of mathematical auxiliary fields (which is what they would be if the Newtonian law of motion would be well-defined) to fields that have a certain physical significance in their own right. More precisely, it is the gauge-invariant linear combinations of $\mathbf{d}_k\tilde{\Phi}$ and $\tilde{\mathbf{A}}_k$ on the r.h.s. of (17) which have physical significance. Even if $\tilde{\Phi}$ separates into a sum of N functions $\tilde{\Phi}^{(k)}$ of the N individual generic world-points, the $\tilde{\Phi}^{(k)}$ have significance only at the world-points of the kth particle. The domain for $\tilde{\Phi}^{(k)}$ is the set of all world-points that are a-priori available to the kth particle, which is the kth copy $\mathbb{M}^4_k$ of $\mathbb{M}^4$.

3.4. Extensions to Non-Genuinely Electromagnetic Systems (I)

While in its genuine setting the theory is only about positive and negative electrons and the electromagnetic fields, so-called *external charges* of physical importance are readily accommodated, as are electromagnetically non-interacting particles. Since the theory aims at providing the classical stepping stone on our way to a more fundamental physical theory, only extensions necessary to mimic such microscopically relevant situations will be considered.[26] Macroscopic reformulations of the theory with continuum charge and current densities, while also possible, are *not* necessary and should be derivable from the microscopic theory as expounded here, to the extent that it is correct, with the help of some law of large numbers.

The most important special case of this generalization is a typical system of many negative electrons and many positive nuclei. Therefore, consider a system of N positive, neutral, or negative point charges, carrying (integer) multiples of the unit charge, and possibly more massive than the electrons. Multiple-units charges are accommodated by simply letting $z_k \in \mathbb{Z}$ now denote the charge of the kth particle (note that the particle may be neutral). If it is desired, the finite size of the nuclei can also be modeled with form factors that have to be put into the theory by hand, but for the sake of simplicity we will not do this here (see refs. 2 and 3 for how to treat extended, spinning charge distribution covariantly in Lorentz electrodynamics.) Heavier masses than the electron's can simply be accommodated by numerical parameters κ_k, expressing the ratio of the electron's to the kth

[26] A derivation of such extensions should be possible in terms of the more fundamental theory.

particle's rest mass, which enter the covariant guiding law for the kth history as multipliers of the coupling constant,

$$\mathbf{u}_k = \mathbf{d}_k\tilde{\Phi} - \kappa_k z_k \alpha \tilde{\mathbf{A}}_k. \tag{21}$$

Note that $\kappa_k \leqslant 1$ if $|z_k| = 1$, and $0 < \kappa_k \ll 1$ if $|z_k| \neq 1$, thereby reducing the influence of the electromagnetic connection on the point history.

4. THE EVOLUTIONARY FORMALISM

4.1. Space and Time; Electricity and Magnetism

Space-like foliations of spacetime have to satisfy[49, 30] the first and second variational equations for the evolution of the first and second fundamental forms, which is constrained by the Gauss and Gauss–Codacci equations. However, most of the theory can be formulated in the familiar space and time splitting of the standard foliation, to which we turn next.[27]

4.1.1. *The Standard Foliation*

The simplest foliation of $\mathbb{M}^4$, called the standard foliation, consists of affine flat space slices. It is generated by the function $T(\varpi) \equiv -\mathbf{e}_0 \cdot \mathbf{w}$, which has a constant time-like four-gradient, $\mathbf{d}T(\varpi) = -\mathbf{g}(\mathbf{e}_0)$. Here, world-vectors are in $\mathbb{T}_{\varpi_0}(\mathbb{M}^4)$ for $\varpi_0 \in \Sigma_0 \subset \mathbb{M}^4$, with $\mathbf{e}_0$ the constant time-like unit normal to Σ_0. Since Σ_0 is then identifiable with its own tangent space, $\Sigma_0 \cong \mathbb{R}^3$, we can globally coordinatize Σ_0 by the restriction to Σ_0 of the global coordinatization of $\mathbb{M}^4$ w.r.t. the Lorentz frame $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at ϖ_0. We identify $\varpi \cong (s^0, s^1, s^2, s^3)$. In these coordinates, we have $g = \operatorname{diag}(1, 1, 1)$ and the spacetime metric is given by the world-line element

$$(\mathrm{d}s)^2 = -(\mathrm{d}s^0)^2 + \sum_{1 \leqslant m \leqslant 3} (\mathrm{d}s^m)^2. \tag{22}$$

If we want to emphasize the "time" and "space" coordinatization of ϖ we write $\varpi \cong (t, \mathbf{s})$, where $t \in \mathbb{R}$ denotes an "instant of time" and $\mathbf{s} \in \Sigma_t$ a "space point," the bold font used for $\mathbf{s}$ indicating that, in the standard foliation, $\Sigma_t \equiv \mathbb{R}^3$ is a vector space.

[27] Constructing suitable foliations will be of prime importance when formulating the Cauchy problem for the general-relativistic extension of our special-relativistic electromagnetic theory, which we already anticipated can be done. As to the importance of selecting good foliations for the Cauchy problem of the general-relativistic vacuum Einstein equations, see ref. 30.

4.1.2. Electric and Magnetic Decomposition of Electromagnetism

In a local basis of one-forms dual to a Lorentz frame $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at $\varpi \in \mathbb{M}^4$, the various electromagnetic p forms read: $\mathbf{A} = A_\mu \, \mathbf{d}s^\mu$ and $\mathbf{F} = F_{\mu\nu} \, \mathbf{d}s^\mu \otimes \mathbf{d}s^\nu$, with $F_{\mu\nu} = -F_{\nu\mu}$. Written with a wedge product this becomes $\mathbf{F} = \frac{1}{2} F_{\mu\nu} \, \mathbf{d}s^\mu \wedge \mathbf{d}s^\nu$. Similarly, $\mathbf{M} = M_{\mu\nu} \, \mathbf{d}s^\mu \otimes \mathbf{d}s^\nu$, with $M_{\mu\nu} = -M_{\nu\mu}$; hence, $\mathbf{M} = \frac{1}{2} M_{\mu\nu} \, \mathbf{d}s^\mu \wedge \mathbf{d}s^\nu$. Finally, $\mathbf{J} = \frac{1}{3!} j^\kappa \varepsilon_{\kappa\lambda\mu\nu} \, \mathbf{d}s^\lambda \wedge \mathbf{d}s^\mu \wedge \mathbf{d}s^\nu$, where $\varepsilon_{\kappa\lambda\mu\nu}$ is the totally antisymmetric symbol with $\varepsilon_{0123} = 1$. The Hodge dual of $\mathbf{J}$ is the one-form ${}^\star\mathbf{J} = j_\kappa \, \mathbf{d}s^\kappa$.

The electric and magnetic decomposition of the electromagnetic connection $\mathbf{A}$ w.r.t. the space and time decomposition gives the electric potential $A = A^0$ and the magnetic vector-potential $\mathbf{A} = (A^1, A^2, A^3)$. The decomposition of the Faraday tensor $\mathbf{F}$ gives us $\mathbf{F} = -E_k \, \mathbf{d}t \wedge \mathbf{d}s^k + B_1 \, \mathbf{d}s^2 \wedge \mathbf{d}s^3 + B_2 \, \mathbf{d}s^3 \wedge \mathbf{d}s^1 + B_3 \, \mathbf{d}s^1 \wedge \mathbf{d}s^2$, where $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ are the electric, respectively magnetic induction, fields; cf. ref. 78. Similarly, the decomposition of the Maxwell tensor $\mathbf{M}$ gives us $\mathbf{M} = H_k \, \mathbf{d}t \wedge \mathbf{d}s^k + D_1 \, \mathbf{d}s^2 \wedge \mathbf{d}s^3 + D_2 \, \mathbf{d}s^3 \wedge \mathbf{d}s^1 + D_3 \, \mathbf{d}s^1 \wedge \mathbf{d}s^2$, where $\mathbf{D} = (D^1, D^2, D^3)$ and $\mathbf{H} = (H^1, H^2, H^3)$ are the electric displacement, respectively the magnetic, vector fields. The decomposition of the electromagnetic current density three-form gives the charge density $j = j^0$ and the current vector-density $\boldsymbol{j} = (j^1, j^2, j^3)$.

4.2. The Space and Time Decomposition of the Actual Field Equations

We now list the geometrical equations in their space and time decomposition, which brings out the evolutionary and the constraint aspects of the theory. In principle, the evolution and constraint equations for the space-like foliations of $\mathbb{M}^4$ will have to be listed among the fundamental equations for the theory. However, since for our Minkowski spacetime we will not require nonstandard foliations, we relegate the discussion of the foliation equations to a later paper. Having stressed this, we now move on with our development of the evolutionary formalism in the standard foliation. In the following, partial derivative w.r.t. time is denoted by the symbol ∂, while ∇ denotes the usual space gradient operator w.r.t. the standard foliation.

4.2.1. The Continuity Equation for Systems of Electric Point Charges

It is readily shown (e.g., ref. 101) that the space and time decomposition of (11) into the microscopic charge "density" $j(t, s)$ and current "vector-density" $\boldsymbol{j}(t, \boldsymbol{s})$ for a system of $N^+ \geqslant 0$ positive and $N^- \geqslant 0$ negative unit point charges reads

$$j(t, \boldsymbol{s}) = \sum_{k \in \mathcal{N}} z_k \, \delta_{\boldsymbol{r}_k(t)}(\boldsymbol{s}) \tag{23}$$

$$\boldsymbol{j}(t, \boldsymbol{s}) = \sum_{k \in \mathcal{N}} z_k \, \delta_{\boldsymbol{r}_k(t)}(\boldsymbol{s}) \, \dot{\boldsymbol{r}}_k(t), \tag{24}$$

where $\delta_{\boldsymbol{r}_k(t)}(\boldsymbol{s})$ denotes the Dirac probability distribution of $\boldsymbol{s} \in \mathbb{R}^3$ concentrated at the position $\boldsymbol{s} = \boldsymbol{r}_k(t) \in \mathbb{R}^3$, and moving with the linear velocity $\frac{d\boldsymbol{s}}{dt} = \dot{\boldsymbol{r}}_k(t)$, at foliation time t. These measure-valued expressions for j and $\boldsymbol{j}$ satisfy, in a weak sense, the familiar continuity equation

$$\partial j + \nabla \cdot \boldsymbol{j} = 0, \tag{25}$$

which expresses the law of charge conservation (10) in space and time decomposition.

4.2.2. Maxwell's Electromagnetic Field Equations

We now address the spacetime between the particle world-lines. The standard space and time decompositions of the Faraday–Maxwell law and the Ampére–Coulomb–Maxwell law yield the general Maxwell equations of the classical electromagnetic fields, familiar from Maxwell's theory of electromagnetic fields in a medium with nontrivial dielectric and magnetic permeability properties and listed in many textbooks, e.g., in refs. 57, 68, and 81. Here of course the "medium" is the nonlinear aether itself. The dynamical variables are the field vectors $\boldsymbol{B}$ and $\boldsymbol{D}$ at each space point $\boldsymbol{s}$, the locations of the point charges excepted.

The evolution of $\boldsymbol{B}$ is governed by

$$\partial \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{26}$$

and constrained by

$$\nabla \cdot \boldsymbol{B} = 0, \tag{27}$$

while the evolution of $\boldsymbol{D}$ is governed by

$$\partial \boldsymbol{D} = \nabla \times \boldsymbol{H} - 4\pi \boldsymbol{j}, \tag{28}$$

and constrained by

$$\nabla \cdot \boldsymbol{D} = 4\pi j. \tag{29}$$

4.2.3. Born and Infeld's Electromagnetic Aether Laws

The electric and magnetic decomposition of the aether laws of Born and Infeld[(21)] defines the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ locally in terms of the natural evolutionary variables electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$,

$$\boldsymbol{E} = \frac{\boldsymbol{D} - \beta^4 \boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{D})}{\sqrt{1 + \beta^4(|\boldsymbol{B}|^2 + |\boldsymbol{D}|^2) + \beta^8\, |\boldsymbol{B} \times \boldsymbol{D}|^2}} \tag{30}$$

$$\boldsymbol{H} = \frac{\boldsymbol{B} - \beta^4 \boldsymbol{D} \times (\boldsymbol{D} \times \boldsymbol{B})}{\sqrt{1 + \beta^4(|\boldsymbol{B}|^2 + |\boldsymbol{D}|^2) + \beta^8\, |\boldsymbol{B} \times \boldsymbol{D}|^2}}, \tag{31}$$

for $\beta \in (0, \infty)$. The notion of weak-field limit is intrinsically well-defined for the Born–Infeld aether laws, in which limit they reduce to Maxwell's pure aether laws

$$\boldsymbol{E} \sim \boldsymbol{D} \qquad \text{(weak field limit)} \tag{32}$$

$$\boldsymbol{H} \sim \boldsymbol{B} \qquad \text{(weak field limit).} \tag{33}$$

We remark that in the formal limit $\beta \to 0$, the Born–Infeld aether laws yield precisely Maxwell's laws of the pure aether, $\boldsymbol{E} = \boldsymbol{D}$ and $\boldsymbol{H} = \boldsymbol{B}$. On the other hand, the limit $\beta \to \infty$ of the Born–Infeld aether laws yields *ultra*[28] Born–Infeld laws

$$\boldsymbol{E} = \frac{\boldsymbol{B} \times \boldsymbol{D}}{|\boldsymbol{B} \times \boldsymbol{D}|} \times \boldsymbol{B} \tag{34}$$

$$\boldsymbol{H} = \frac{\boldsymbol{D} \times \boldsymbol{B}}{|\boldsymbol{D} \times \boldsymbol{B}|} \times \boldsymbol{D}, \tag{35}$$

unless $\boldsymbol{B} \times \boldsymbol{D} = \boldsymbol{0}$, in which case we find $\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{H} = \boldsymbol{0}$. Completing the source-free Maxwell field equations with either the Maxwell laws of the pure aether or with the ultra Born–Infeld laws of the aether result in field theories which are invariant under the full conformal group of Minkowski space. The source-free Maxwell field equations for the Maxwell laws of the pure aether have of course been exhaustively studied, e.g., refs. 57, 68, and 81. The source-free Maxwell field equations for the ultra Born–Infeld laws of the aether are studied in ref. 11, 12, and 24.

[28] This coinage was proposed in ref. 11.

4.2.4. The Laws for the Electric and Magnetic Potentials

While $\mathbf{F}$ is defined as $\mathbf{F} = \mathbf{d}A$, this equation at the same time gives the status of a primitive dynamical field variable to the magnetic vector potential $\boldsymbol{A}$, with the electric field strength $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ acting as source terms in the evolution, respectively constraint equations for $\boldsymbol{A}$. Thus, the evolution of $\boldsymbol{A}$ is governed by

$$\partial \boldsymbol{A} = -\nabla A - \boldsymbol{E} \tag{36}$$

and constrained by

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}. \tag{37}$$

The electric potential A on the other hand acquires a dynamical status only within a Poincaré-invariant gauge. For instance, in the V. Lorenz–H. A. Lorentz gauge[49, 58, 59] the evolution of A is governed by

$$\partial A = -\nabla \cdot \boldsymbol{A} \tag{38}$$

and not constrained by any other equation. However, while in the Lorenz–Lorentz gauge the Maxwell–Lorentz equations with prescribed (point) sources are equivalent to a decoupled set of non-homogeneous wave equations for A, $\boldsymbol{A}$ that are readily solved by the Liénard–Wiechert potentials,[72, 106] this gauge achieves no such simplification for the Maxwell–Born–Infeld equations with (point) sources. Hence, while for the sake of concreteness we shall work in the Lorenz–Lorentz gauge one might as well look for a more convenient Poincaré-invariant gauge to work in.

4.3. The *t*-Synchronized Proper Many-Body Hamilton–Jacobi Formalism

4.3.1. t-Synchronized Ordered Configurations

We begin by defining the (foliation time) t-synchronized configuration space of N ordered space-points, which consists of ordered $3N$-tupels $\boldsymbol{S} \equiv (\boldsymbol{s}_1,..., \boldsymbol{s}_N) \in \mathbb{R}^{3N}_{\neq}$; here, $\mathbb{R}^{3N}_{\neq} \equiv \mathbb{R}^{3N} \setminus \{\boldsymbol{S}: \boldsymbol{s}_k = \boldsymbol{s}_l \text{ for some } k \neq l\}$, i.e., coincidence points are removed. The N individual point charge trajectories $t \mapsto \boldsymbol{s}|_{H_k} = \boldsymbol{r}_k(t) \in \mathbb{R}^3$ now correspond to a single trajectory $t \mapsto \boldsymbol{S} = \mathbf{R}(t) \in \mathbb{R}^{3N}_{\neq}$ in this ordered configuration space, where $\mathbf{R}(t) = (\boldsymbol{r}_1(t),..., \boldsymbol{r}_N(t))$. Our goal is a Hamilton–Jacobi law of evolution such that $\dot{\mathbf{R}}(t) = \mathbf{V}(t, \mathbf{R}(t))$ for a velocity flow field $\mathbf{V}(t, \boldsymbol{S}) \equiv (\boldsymbol{v}_1,..., \boldsymbol{v}_N)(t, \boldsymbol{S})$ on $\mathbb{R}^{3N}_{\neq}$ obtained with the phase function $\Phi(t, \boldsymbol{S})$ on $\mathbb{R}^{3N}_{\neq}$ satisfying a Hamilton–Jacobi PDE.

4.3.2. The t-Synchronized Equations of the Space and Time Decomposition of the ♯Fields

The space and time decomposition of ${}^{\sharp}\boldsymbol{A}(\varpi, \varpi_1, \dots, \varpi_N)$ into components ${}^{\sharp}A$ and ${}^{\sharp}\boldsymbol{A}$ and subsequent t-synchronization gives us the fields

$$A^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) \equiv {}^{\sharp}A(t, \boldsymbol{s}, t_1, \boldsymbol{s}_1, \dots, t_k, \boldsymbol{s}_k, \dots, t_N, \boldsymbol{s}_N)\big|_{t_1 = t_2 = \cdots = t_N = t}, \tag{39}$$

$$\boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) \equiv {}^{\sharp}\boldsymbol{A}(t, \boldsymbol{s}, t_1, \boldsymbol{s}_1, \dots, t_k, \boldsymbol{s}_k, \dots, t_N, \boldsymbol{s}_N)\big|_{t_1 = t_2 = \cdots = t_N = t} \tag{40}$$

on $\mathbb{R}^{3(N+1)}$ (etc. for the other ♯fields). As stipulated earlier, by conditioning with the actual configuration we want to obtain the actual fields on $\mathbb{M}^4$ (in Lorentz gauge, say), i.e., $A^{\sharp}(t, \boldsymbol{s}, \mathbf{R}(t)) = A(t, \boldsymbol{s})$ and $\boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \mathbf{R}(t)) = \boldsymbol{A}(t, \boldsymbol{s})$ (etc.). This canonically fixes the equations for the t-synchronized space and time decomposition of the ♯fields. Namely, $A^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$, $\boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$, and $\boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ satisfy the evolution equations

$$\partial A^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = -\mathbf{V}(t, \boldsymbol{S}) \cdot \nabla_{\boldsymbol{S}} A^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) - \nabla \cdot \boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}), \tag{41}$$

$$\partial \boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = -\mathbf{V}(t, \boldsymbol{S}) \cdot \nabla_{\boldsymbol{S}} \boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) - \nabla A^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) - \boldsymbol{E}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}), \tag{42}$$

$$\partial \boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = -\mathbf{V}(t, \boldsymbol{S}) \cdot \nabla_{\boldsymbol{S}} \boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) + \nabla \times \boldsymbol{H}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) - 4\pi \boldsymbol{j}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}), \tag{43}$$

where $\mathbf{V}(t, \boldsymbol{S}) \cdot \nabla_{\boldsymbol{S}} \equiv \sum_{k=1}^{N} \boldsymbol{v}_k(t, \boldsymbol{S}) \cdot \nabla_k$; furthermore, $\boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ obeys the constraint equation

$$\nabla \cdot \boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = 4\pi j^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}), \tag{44}$$

where[29]

$$j^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = \sum_{k \in \mathcal{N}} z_k \, \delta_{\boldsymbol{s}_k}(\boldsymbol{s}), \tag{45}$$

$$\boldsymbol{j}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = \sum_{k \in \mathcal{N}} z_k \, \delta_{\boldsymbol{s}_k}(\boldsymbol{s}) \, \boldsymbol{v}_k(t, \boldsymbol{S}). \tag{46}$$

The fields $\boldsymbol{E}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ and $\boldsymbol{H}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ in (42), (43) are defined in terms of $\boldsymbol{D}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ and $\boldsymbol{B}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ in precisely the same manner as the actual fields $\boldsymbol{E}(t, \boldsymbol{s})$ and $\boldsymbol{H}(t, \boldsymbol{s})$ are defined in terms of $\boldsymbol{D}(t, \boldsymbol{s})$ and $\boldsymbol{B}(t, \boldsymbol{s})$ through the Born–Infeld aether laws (30), (31), while $\boldsymbol{B}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ in turn is defined in terms of $\boldsymbol{A}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$ in precisely the same manner as the actual $\boldsymbol{B}(t, \boldsymbol{s})$ is defined in terms of the actual $\boldsymbol{A}(t, \boldsymbol{s})$ in (37).

It is straightforward to verify that by substituting the actual configuration $\mathbf{R}(t)$ for the generic $\boldsymbol{S}$ in the t-synchronized ♯fields satisfying the

[29] Notice that $j^{\sharp}$ is actually t-independent. Notice furthermore that $\nabla_{\boldsymbol{s}_k} \delta_{\boldsymbol{s}_k}(\boldsymbol{s}) = -\nabla_{\boldsymbol{s}} \delta_{\boldsymbol{s}_k}(\boldsymbol{s})$. Hence, the ♯field re-formulation of the continuity equation for the charge conservation (in spacetime), $\partial j^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) = -\mathbf{V}(t, \boldsymbol{S}) \cdot \nabla_{\boldsymbol{S}} j^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S}) - \nabla \cdot \boldsymbol{j}^{\sharp}(t, \boldsymbol{s}, \boldsymbol{S})$, is an identity, not an independent equation.

above equations, we obtain the actual electromagnetic potentials, fields, and charge-current densities satisfying the Maxwell–Born–Infeld field Equations (in Lorentz–Lorenz gauge).

4.3.3. The t-Synchronized Hamilton–Jacobi Equations

By conditioning $A^\sharp(t, s, \boldsymbol{S})$ and $\boldsymbol{A}^\sharp(t, s, \boldsymbol{S})$ with $\boldsymbol{s} = \boldsymbol{s}_k$ for each $k = 1,\dots, N$, we now obtain the t-synchronized $\tilde{A}_k$ and $\tilde{\boldsymbol{A}}_k$ (etc.) fields on $\mathbb{R}^{3N}_{\neq}$, for which we write

$$A_k(t, \boldsymbol{S}) \equiv \tilde{A}_k(t_1, \boldsymbol{s}_1,\dots, t_N, \boldsymbol{s}_N)|_{t_1 = t_2 = \cdots = t_N = t} \tag{47}$$

and

$$\boldsymbol{A}_k(t, \boldsymbol{S}) \equiv \tilde{\boldsymbol{A}}_k(t_1, \boldsymbol{s}_1,\dots, t_N, \boldsymbol{s}_N)|_{t_1 = t_2 = \cdots = t_N = t}. \tag{48}$$

We shall now show that the space and time decomposition of the geometric law for each point-history (17) is equivalent to a Hamilton–Jacobi law of point charge motion. We consider the genuinely electromagnetic setting first. The non-genuinely electromagnetic extensions are given subsequently.

A Single Positive or Negative Unit Point Charge

When there is only a single point charge, in space and time decomposition (no synchronization necessary now; i.e., $t_1 = t$ automatically), we can immediately identify $\tilde{A}_1(\varpi_1) \equiv A_1(t, \boldsymbol{s}_1)$ and $\tilde{\boldsymbol{A}}_1(\varpi_1) \equiv \boldsymbol{A}_1(t, \boldsymbol{s}_1)$; furthermore, we clearly have $\tilde{\Phi}_1(\varpi_1) \equiv \Phi(t, \boldsymbol{s}_1)$, where $\Phi\colon \mathbb{R}^{1,3} \to \mathbb{R}$ is the scalar phase function associated with the point-history of the only electron in that world. In terms of Φ we now define a Minkowski velocity field on $\mathbb{M}^4_1$, given by the system of equations

$$u(t, \boldsymbol{s}_1) = -\partial\Phi(t, \boldsymbol{s}_1) - \pm\alpha A_1(t, \boldsymbol{s}_1), \tag{49}$$

$$\boldsymbol{u}(t, \boldsymbol{s}_1) = \nabla_1\Phi(t, \boldsymbol{s}_1) - \pm\alpha \boldsymbol{A}_1(t, \boldsymbol{s}_1). \tag{50}$$

The condition that the l.h.s. be a Minkowski velocity vector for each value of its arguments gives the space and time decomposition of the geometrical law (18) for the phase function Φ,

$$-(\partial\Phi + \pm\alpha A_1)^2 + |\nabla_1\Phi - \pm\alpha\boldsymbol{A}_1|^2 = -1, \tag{51}$$

and the future-orientation of $\boldsymbol{u}$ selects the following root of (51),

$$\partial\Phi = -\sqrt{1 + |\nabla_1\Phi - \pm\alpha\boldsymbol{A}_1|^2} - \pm\alpha A_1. \tag{52}$$

The appearance of (51), and of (52), is familiar from the relativistic Hamilton–Jacobi equation for a single point charge interacting with external electromagnetic fields;[68] cf. our section on test particles Hamilton–Jacobi theory. Appearances are however misleading, for here the fields are the total fields.

The geometric law (17) for the actual history H of the single negative or positive point electron is then the system of guiding equations

$$\left.\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|_H = u(t, \mathbf{s}_1)|_H \tag{53}$$

$$\left.\frac{\mathrm{d}\mathbf{s}_1}{\mathrm{d}\tau}\right|_H = \mathbf{u}(t, \mathbf{s}_1)|_H. \tag{54}$$

We finally eliminate τ in favor of our foliation time t to get the equation for $t \mapsto \mathbf{s}_1$. Thus, $\tau \mapsto (t, \mathbf{s}_1) = (w_1(\tau), \mathbf{w}_1(\tau))$ is replaced by $t \mapsto (t, \mathbf{s}_1) = (r_1(t), \mathbf{r}_1(t))$, whence $r(.) \equiv \mathrm{id}(.)$ (we would not need to keep the index $_1$, but for the sake of clarity we do). The change of the derivatives is done according to the familiar relativistic formula $\mathrm{d}\tau = \sqrt{1 - |\dot{\mathbf{r}}_1|^2}\,\mathrm{d}t$, where $\dot{\mathbf{r}}_1(t)$ is the conventional velocity of the particle at $\mathbf{s}_1 = \mathbf{r}_1(t)$ in the standard-foliation space $\mathbb{R}^3$. Hence, for a single point charge coupled to the electromagnetic Maxwell–Born–Infeld fields, we can finally rewrite the space-part of the guiding equation into

$$\left.\frac{\mathrm{d}\mathbf{s}_1}{\mathrm{d}t}\right|_H = \left.\frac{\nabla_1 \Phi(t, \mathbf{s}_1) - \pm\alpha \mathbf{A}_1(t, \mathbf{s}_1)}{\sqrt{1 + |\nabla_1 \Phi(t, \mathbf{s}_1) - \pm\alpha \mathbf{A}_1(t, \mathbf{s}_1)|^2}}\right|_H. \tag{55}$$

Lastly, we note that substitution of the actual position $\mathbf{r}_1(t)$ for the generic $\mathbf{s}_1$ in $A_1(t, \mathbf{s}_1)$ and $\mathbf{A}_1(t, \mathbf{s}_1)$ gives the actual potential fields evaluated at the location of the single point charge, i.e., we have $A_1(t, \mathbf{r}_1(t)) = A(t, \mathbf{r}_1(t))$ and $\mathbf{A}_1(t, \mathbf{r}_1(t)) = \mathbf{A}(t, \mathbf{r}_1(t))$. Thus, for the actual solution $t \mapsto \mathbf{s}_1 = \mathbf{r}_1(t)$ of (55), we find the identity

$$\dot{\mathbf{r}}_1(t) = \frac{\nabla_1 \Phi(t, \mathbf{r}_1(t)) - \pm\alpha \mathbf{A}(t, \mathbf{r}_1(t))}{\sqrt{1 + |\nabla_1 \Phi(t, \mathbf{r}_1(t)) - \pm\alpha \mathbf{A}(t, \mathbf{r}_1(t))|^2}}, \tag{56}$$

which is precisely (14), our point of departure. We have come full circle.

Many Positive and Negative Unit Point Charges

For N (negative and positive) unit point charges, space and time decomposition gives $\tilde{\Phi}(\varpi_1, \ldots, \varpi_N) = \tilde{\Phi}(t_1, \mathbf{s}_1, \ldots, t_N, \mathbf{s}_N)$, and synchronization gives us the *t-synchronized configuration space phase function*

$\Phi(t, \boldsymbol{S}) \equiv \tilde{\Phi}(t, \boldsymbol{s}_1, \ldots, t, \boldsymbol{s}_N)$, with the help of which we now define a Minkowski velocities field on $\mathbb{R} \times \mathbb{R}^{3N}_{\neq}$, the kth component of which is given by the system of equations

$$u_k(t, \boldsymbol{S}) = -\partial_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) - z_k \alpha A_k(t, \boldsymbol{S}) \tag{57}$$

$$\boldsymbol{u}_k(t, \boldsymbol{S}) = \nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S}). \tag{58}$$

The constraint that the left-hand sides of (57) and (58) are the components in the kth copy $\mathbb{M}^4_k$ of $\mathbb{M}^4$ of a future-oriented Minkowski-velocities vector field on configuration space gives a Hamilton–Jacobi type partial differential equation on $\mathbb{R} \times \mathbb{R}^{3N}_{\neq}$,

$$\partial_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) = -\sqrt{1 + |\nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})|^2} - z_k \alpha A_k(t, \boldsymbol{S}). \tag{59}$$

Using now $\mathrm{d}\tau = \sqrt{1 - |\dot{\boldsymbol{r}}_k|^2}\, \mathrm{d}t$ for each k gives us the Hamilton–Jacobi type guiding equation for the actual motion of the kth point-charge along H_k,

$$\left.\frac{\mathrm{d}\boldsymbol{s}_k}{\mathrm{d}t}\right|_{H_k} = \left.\frac{\nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})}{\sqrt{1 + |\nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{s}_k, \ldots) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})|^2}}\right|_{H_k}. \tag{60}$$

Finally, since substitution of the actual configuration $\boldsymbol{R}(t)$ for the generic $\boldsymbol{S}$ in $A_k(t, \boldsymbol{S})$ and $\boldsymbol{A}_k(t, \boldsymbol{S})$ gives the actual potential fields evaluated at the location of the kth point charge, i.e., since $A_k(t, \mathbf{R}(t)) = \tilde{A}_k(t, \boldsymbol{r}_1(t), \ldots, t, \boldsymbol{r}_N(t)) = A(t, \boldsymbol{r}_k(t))$ and $\boldsymbol{A}_k(t, \mathbf{R}(t)) = \tilde{\boldsymbol{A}}_k(t, \boldsymbol{r}_1(t), \ldots, t, \boldsymbol{r}_N(t)) = \boldsymbol{A}(t, \boldsymbol{r}_k(t))$, (60) turns into the identity

$$\dot{\boldsymbol{r}}_k(t) = \frac{\nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{r}_k(t), \ldots) - z_k \alpha \boldsymbol{A}(t, \boldsymbol{r}_k(t))}{\sqrt{1 + |\nabla_k \tilde{\Phi}(\ldots, t, \boldsymbol{r}_k(t), \ldots) - z_k \alpha \boldsymbol{A}(t, \boldsymbol{r}_k(t))|^2}}, \tag{61}$$

which is (14). Once again we have completed the loop back to our point of departure, though not quite. We still have to state the results in terms of $\Phi(t, \boldsymbol{S})$ satisfying a single Hamilton–Jacobi PDE. But, clearly, $\Phi(t, \boldsymbol{S})$ satisfies

$$\partial \Phi = -\sum_{k \in \mathcal{N}} \left(\sqrt{1 + |\nabla_k \Phi - z_k \alpha \boldsymbol{A}_k|^2} + z_k \alpha A_k\right), \tag{62}$$

and (60) can be rephrased as

$$\left.\frac{\mathrm{d}\boldsymbol{s}_k}{\mathrm{d}t}\right|_{H_k} = \left.\frac{\nabla_k \Phi(t, \boldsymbol{S}) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})}{\sqrt{1 + |\nabla_k \Phi(t, \boldsymbol{S}) - z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})|^2}}\right|_{H_k}. \tag{63}$$

We finally remark that in the decoherent case variables can even be separated in Φ as $\Phi = \sum_k \Phi^{(k)}$, so that the kth point-history has associated with it an individual scalar phase function $\Phi^{(k)}: \mathbb{R}_k^{1,3} \to \mathbb{R}$.

Extensions to Non-Genuinely Electromagnetic Systems (II)

We here briefly summarize how the extensions to non-genuinely electromagnetic settings affect the equations of the evolutionary formulation of the theory. The effect that these modified formulas have on the conservation laws and variational principles can then readily be worked out.

In the fairly general microscopic setting presented in Section 3.4, the change of value space for the z_k from $\{-1, 1\}$ to $\mathbb{Z}$ does not entail any change in the formulas. However, the kth component of configuration space trajectory $\boldsymbol{S} = \mathbf{R}(t)$, where $\boldsymbol{S} \equiv (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N)$, now satisfies the Hamilton–Jacobi guiding law

$$\left.\frac{\mathrm{d}\boldsymbol{s}_k}{\mathrm{d}t}\right|_{H_k} = \left.\frac{\nabla_k \Phi(t, \boldsymbol{S}) - \kappa_k z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})}{\sqrt{1 + |\nabla_k \Phi(t, \boldsymbol{S}) - \kappa_k z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})|^2}}\right|_{H_k} \tag{64}$$

where again $\kappa_k \leqslant 1$ is the ratio of the electron's to kth particle's rest mass, and the subscripts at the operators indicate once again with respect to which of the particle co-ordinates to take the derivative. The Hamilton–Jacobi equation on t-synchronized $3N$-dimensional configuration space becomes

$$\partial \Phi(t, \boldsymbol{S}) = -\sum_{k \in \mathcal{N}} \left(\sqrt{1 + |\nabla_k \Phi(t, \boldsymbol{S}) - \kappa_k z_k \alpha \boldsymbol{A}_k(t, \boldsymbol{S})|^2} + \kappa_k z_k \alpha A_k(t, \boldsymbol{S}) \right). \tag{65}$$

While it is obvious from the generality of the above formulas that the electromagnetic effects of particles and antiparticles are modeled on an equal footing, the primary application of the theory should be to normal matter modeled by N^+ positive point charges of various sorts, and N^- negative unit point charges, representing a system of nuclei and electrons. The most sensible application should be to high-temperature plasma, but to the extent that quantum effects are negligible, applications to fluids, solids, and even molecules and atoms may be possible. For $N^- > 1$ we may wonder whether we can even impose the Pauli principle for the electrons already at the classical level; we come to this point in a moment.

The Born–Oppenheimer Approximation

We remark that the Born–Oppenheimer approximation, in which the nuclei are assumed to be infinitely massive, can also easily be implemented by formally letting $\kappa_k \downarrow 0$ for the positive charges representing nuclei in our model. Inspection of (64) reveals that $\kappa_k \downarrow 0$ gives for each nucleus the equation of a free straight motion, as it should be the case. This simplifies the dynamical problem for the charges to solving a reduced Hamilton–Jacobi equation for the N^- electrons, which is coupled to the nuclei only via the electromagnetic potentials. Furthermore, if there is only one nucleus, then by performing a Lorentz boost and a translation one can assume that the nucleus is permanently at rest at the origin.

4.3.4. Gauge Invariance (Part II)

Since the $A^\sharp$ and $\boldsymbol{A}^\sharp$ inherit the gauge transformations from A and $\boldsymbol{A}$, we see that Eqs. (36), (37), (38) for the electric and magnetic potentials, and Eqs. (62), (63) for the point charge motions, are invariant under gauge transformations

$$\Phi(t, \boldsymbol{S}) \to \Phi(t, \boldsymbol{S}) + \alpha \sum_{k \in \mathcal{N}} z_k \Upsilon(t, \boldsymbol{s}_k), \tag{66}$$

$$A_k(t, \boldsymbol{S}) \to A_k(t, \boldsymbol{S}) - \partial \Upsilon(t, \boldsymbol{s}_k), \tag{67}$$

$$\boldsymbol{A}_k(t, \boldsymbol{S}) \to \boldsymbol{A}_k(t, \boldsymbol{S}) + (\nabla \Upsilon)(t, \boldsymbol{s}_k), \tag{68}$$

with any relativistic scalar field $\Upsilon\colon \mathbb{R}^{1,3} \to \mathbb{R}$ satisfying the wave equation

$$\partial^2 \Upsilon = \nabla^2 \Upsilon \tag{69}$$

with $\nabla^2 = \Delta$, the Laplacian on $\mathbb{R}^3$. Since a (sufficiently regular) solution of (69) in $\mathbb{R}_+ \times \mathbb{R}^3$ is uniquely determined by the initial data for Υ and its time derivative $\partial \Upsilon$, the gauge freedom that is left concerns the initial conditions of Φ, A, $\boldsymbol{A}$.

4.4. The Cauchy Problem for the Physical State

After having recast the geometrical laws of electromagnetism into the evolutionary formalism w.r.t. the standard foliation of Minkowski space-time, we here collect all the variables for which an evolution equation in t has been obtained: the actual world variables $A(.,\boldsymbol{s})$, $\boldsymbol{A}(.,\boldsymbol{s})$, $\boldsymbol{B}(.,\boldsymbol{s})$, $\boldsymbol{D}(.,\boldsymbol{s})$, $j(.,\boldsymbol{s})$, $\mathbf{R}(.)$, as well as $\Upsilon(.,\boldsymbol{s})$ and $\partial\Upsilon(.,\boldsymbol{s})$, and the Hamilton–Jacobi variables $A^\sharp(.,\boldsymbol{s},\boldsymbol{S})$, $\boldsymbol{A}^\sharp(.,\boldsymbol{s},\boldsymbol{S})$, $\boldsymbol{D}^\sharp(.,\boldsymbol{s},\boldsymbol{S})$, $\Phi(.,\boldsymbol{S})$. However, the information content in the listed variables is partly redundant. Indeed, in the configuration space setting we already cleaned up this redundancy, but there is a reason why we left the redundancy of the actual world variables.

Namely, which variables most conveniently represent the physical evolutionary degrees of freedom,[30] depends on whether source-free evolutions or those with point charge sources are considered.

4.4.1. Source-Free Evolutions

In the absence of any point charge sources, the only evolutionary variables are the solenoidal vector fields $s \mapsto \boldsymbol{B}(.,s)$ and $s \mapsto \boldsymbol{D}(.,s)$. These are in fact canonically conjugate variables for the first-order field evolution Eqs. (26) and (28) without source terms, which constitute a Hamiltonian dynamical system; cf. ref. 11. It is not too difficult to show that sufficiently regular, finite energy solenoidal initial data at $t = 0$ launch a unique evolution locally in time. The much harder problem of global existence is widely open; see however ref. 73 for global existence results for a related scalar equation.[31] Note that the qualifier "solenoidal" for the initial data implies that the constraint Eqs. (27) and (29) are satisfied for all t, as follows by taking the divergence of the evolution Eqs. (26) and (28). Note furthermore that no gauge freedom needs to be taken care of.

4.4.2. Evolutions with Point Charge Sources

The situation is considerably more involved when point charges are present. The actual electromagnetic fields on Minkowski spacetime, $\boldsymbol{B}$ and $\boldsymbol{D}$, lose the status they enjoyed in the charge-free situation. Indeed, to begin with, we sort out the redundant field quantities on actual Minkowski spacetime. In particular, while the electromagnetic potentials A and $\boldsymbol{A}$ now enter, rather than as the constraint equation for $\boldsymbol{A}$ given $\boldsymbol{B}$, (37) ought to be read as defining $\boldsymbol{B}$ given $\boldsymbol{A}$. The constraint Eq. (27) for $\boldsymbol{B}$ is then automatically satisfied; furthermore, by taking the curl of the evolution Eq. (36) for $\boldsymbol{A}$ we see that the evolution Eq. (26) for $\boldsymbol{B}$ is also satisfied if the one for $\boldsymbol{A}$, (36), is. Hence, $\boldsymbol{B}$ can be eliminated from the list of independent field

[30] There are different conventions in the literature as to what constitutes a degree of freedom. Traditioned in Newtonian mechanics one normally refers to the independent second-order (in time) equations as the *dynamical* degrees of freedom (e.g., a Newtonian point particle moving along a line and satisfying Newton's equation of motion with a linear restoring force (a harmonic oscillator) counts as a one-degree-of-freedom dynamical system). However, we here mean the number of data needed to specify the *state* of the physical system on any leaf of the standard foliation, which is then evolved uniquely in time by independent first-order *evolution* equations.

[31] After the original writing of these lines, a proof of the global well-posedness of the source-free Maxwell–Born–Infeld field equations for small initial data patterned after Lindblad's paper[(73)] has been announced.[(26)]

variables on Minkowski spacetime.[32] Working with $\mathbf{A}$ and $\mathbf{D}$, plus A, now gives us some gauge freedom to choose from. By imposing the Lorenz–Lorentz gauge, which yields the evolution Eq. (38) for A, we have already removed some of the gauge freedom, but we can still impose a convenient gauge constraint on part of the initial conditions for Φ, A, $\mathbf{A}$ as long as it is compatible with the physical constraints on Φ, A, $\mathbf{A}$, and to the extent that it can be accommodated uniquely by suitable initial data $\Upsilon(0, \mathbf{s})$ and $\partial\Upsilon(0, \mathbf{s})$ for the wave equation $\Box\Upsilon = 0$. For instance, although perhaps not the most convenient thing to do, it is always possible to demand $A(0, \mathbf{s}) = 0$ and $\nabla \cdot \mathbf{A}(0, \mathbf{s}) = 0$ which fixes Υ up to a harmonic function, still do be disposed of. Yet another evolution equation is redundant, namely (25), for the definitions (23) and (24) of j and $\mathbf{j}$ guarantee that the continuity Eq. (25) is satisfied for all times. This leaves us with the following list of *a priori* independent actual fields on Minkowski spacetime: $\mathbf{D}(\,.\,, \mathbf{s})$, the electric displacement vector at $\mathbf{s}$ away from all $\mathbf{r}_k(.)$; $\mathbf{A}(\,.\,, \mathbf{s})$, the magnetic vector potential at $\mathbf{s}$ away from all $\mathbf{r}_k(.)$; and $A(\,.\,, \mathbf{s})$, the electric potential at $\mathbf{s}$ away from all $\mathbf{r}_k(.)$. However, to compute those fields we need to know the motion of the actual particles configuration, and to compute that one, we actually need to work with a whole parameter family of such $\mathbf{D}$, $\mathbf{A}$, and A fields, namely their corresponding $^\sharp$field cousins, so that the $\mathbf{D}$, $\mathbf{A}$, and A fields themselves are not anymore independent degrees of freedom either. Thus, the evolutionary degrees of freedom comprise:

- the $^\sharp$fields variables
  - $\mathbf{D}^\sharp(\,.\,, \mathbf{s}, \mathbf{S})$, the generalized electric displacement field;
  - $\mathbf{A}^\sharp(\,.\,, \mathbf{s}, \mathbf{S})$, the generalized magnetic potential;
  - $A^\sharp(\,.\,, \mathbf{s}, \mathbf{S})$, the generalized electric potential;
- the Hamilton–Jacobi field variable
  - $\Phi(\,.\,, \mathbf{S})$, the phase function at $\mathbf{S}$, coincidence points excluded;
- and finally, the N-particles' variable
  - $\mathbf{R}(.) = (\mathbf{r}_1(.),\dots, \mathbf{r}_N(.))$, the actual configuration space point;

all constrained by the constraints equations. In regard to our gauge and the space and time decomposition, these variables constitute the classical electromagnetic state Ω^{cl} with point charges. The set of classical states is denoted Γ^{cl}.

[32] We remark that with prescribed point charge sources, $\mathbf{A}$ and $\mathbf{D}$ form a canonically conjugate pair for a Hamiltonian field-dynamical system, in which the constraint of Coulomb's law is incorporated at the expense of a Lagrange parameter field, A. Of course, technically one can work with A, $\mathbf{A}$, and $\mathbf{D}$ also in the absence of point charges, but in that case $\mathbf{A}$ and A would then be considered to be purely auxiliary fields, the primary fields still being $\mathbf{B}$ and $\mathbf{D}$.

The Cauchy problem for $\Omega^{\mathrm{cl}}(t)$ is to solve its system of evolution equations supplemented at initial time, say $t=0$, by the data $\Omega^{\mathrm{cl}}(0)$, which in case of $A^\sharp(0, \boldsymbol{s}, \boldsymbol{S})$, $\boldsymbol{A}^\sharp(0, \boldsymbol{s}, \boldsymbol{S})$, and $\boldsymbol{D}^\sharp(0, \boldsymbol{s}, \boldsymbol{S})$, we demand to satisfy asymptotic vanishing conditions as $|\boldsymbol{s}| \to \infty$ for each fixed $\boldsymbol{S}$. The initial data for the $^\sharp$variables and for Φ have to be chosen such that, if the initial data $\mathbf{R}(0)$ for the actual configuration are substituted for the generic configuration $\boldsymbol{S}$, then the $^\sharp$fields restricted to $\mathbb{R}^3$ become just the actual initial fields, viz. $A^\sharp(0, \boldsymbol{s}, \boldsymbol{R}(0)) = A(0, \boldsymbol{s})$, $\boldsymbol{A}^\sharp(0, \boldsymbol{s}, \boldsymbol{R}(0)) = \boldsymbol{A}(0, \boldsymbol{s})$, and $\boldsymbol{D}^\sharp(0, \boldsymbol{s}, \boldsymbol{R}(0)) = \boldsymbol{D}(0, \boldsymbol{s})$, and the covariant k-gradient of Φ (divided by the obvious relativistic square root term) gives the initial velocity of the kth particle. Otherwise the Hamilton–Jacobi part (inclusive the $^\sharp$fields) does not depend on the actual variables, and in this sense poses an autonomous problem; the actual configuration space trajectory $t \mapsto \boldsymbol{R}(t)$ is solved for subsequently by integrating the N guiding equations obtained from the velocity field generated by Φ. Finally $\boldsymbol{R}(t)$ is substituted for $\boldsymbol{S}$ in the computed $^\sharp$fields to get the actual fields A, $\boldsymbol{A}$ (giving $\boldsymbol{B}$), and $\boldsymbol{D}$ which satisfy the Maxwell–Born–Infeld field equations with actual point charge and current densities according to the motion described by $\boldsymbol{R}(t)$.

We claim that in sharp contrast to the ill-defined Lorentz electrodynamics with point charges, our Cauchy problem can be set up consistently for (classical-)physically generic situations, such that the Hamilton–Jacobi PDE and the $^\sharp$field equations launch Φ and the $^\sharp$fields differentiably into the immediate future of $t=0$, and the guiding equation launches $\boldsymbol{R}$ into its immediate future. More to the point:

Proposition 4.1. The Cauchy problem for Ω^{cl} is well-defined.

Whether the Cauchy problem is also well-posed is a subtler issue. Recall that local well-posedness means that for all initial states $\Omega^{\mathrm{cl}}(0)$ in some sufficiently small open subset of Γ^{cl} there exists a $T>0$ independent of the specific initial state but dependent on the open subset, such that there is a unique evolution $t \mapsto \Omega^{\mathrm{cl}}(t) \in C^0((0,T), \Gamma^{\mathrm{cl}})$, satisfying $\lim_{t\downarrow 0} \Omega^{\mathrm{cl}}(t) = \Omega^{\mathrm{cl}}(0)$. This should be straightforward to prove for a large class of physically relevant data. Global well-posedness on the other hand is a truly hard issue; in fact, we do not expect that the Cauchy problem will be generically globally well-posed. However, as is well-known from the Hamilton–Jacobi PDE in classical non-relativistic mechanics, the evolution may break down also because Φ may lose its single-valuedness. This type of breakdown is akin to coordinate-induced singularities in general relativity that do not correspond to singularities in the curvature of spacetime, and it has to be distinguished from a physical breakdown in finite time. Hard analysis will illuminate the situation in future work.

We will illustrate the Cauchy problem explicitly with two examples of single-particle dynamics: the simplest, static one, which already is too much to handle for the Newtonian law of motion with total electromagnetic fields, and the next simplest one, two oppositely charged point particles initially at rest, one of which is treated in Born–Oppenheimer approximation. However, this we defer to Section 6, after we have collected some pertinent technical materials.

4.4.3. Permutations and t-Synchronized Natural Configurations

Since the labeling of the charges within each species of point charges is ambiguous, the configuration space of N ordered space-points, $\times_{k \in \mathcal{N}} \mathbb{R}^3$, is actually unnatural, though convenient. Following ref. 38 the natural configuration space for N identical particles is the space of finite subsets of $\mathbb{R}^3$ with cardinality N, i.e., $\mathbb{R}^{3N}_{\neq}/S_N$, denoted ${}^N\mathbb{R}^3$. Here, S_N is the symmetric group of N elements. The natural configuration space ${}^N\mathbb{R}^3$ is not simply connected and quite naturally leads to the distinction of bosonic and fermionic wave functions in quantum mechanics.[38] Curiously, the *Pauli principle* for bosons can be implemented already at the classical level by more or less similar reasoning. Thus, realizing $\mathbb{R}^{3N}_{\neq}$ as the universal covering space for ${}^N\mathbb{R}^3$, in the genuinely electromagnetic setting of the theory in which all point charges represent electrons, we can impose the t-synchronized "bosonic" permutation relations on $\mathbb{R}^{3N}_{\neq}$,

$$\Phi(t,..., s_k,..., s_l,...) - \Phi(t,..., s_l,..., s_k,...) = 2\pi Z \tag{70}$$

where $Z \in \mathbb{Z}$; in fact, one may simply have

$$\Phi(t,..., s_k,..., s_l,...) - \Phi(t,..., s_l,..., s_k,...) = 0. \tag{71}$$

This symmetry postulate at time t needs to be preserved under the evolution, which requires that the ${}^\sharp$fields are symmetric under permutations as well. We remark that imposing the Pauli principle does not interfere with the gauge freedom: If Φ satisfies the permutation relations (72) or (70), then so does $\Phi + \alpha \sum_{k \in \mathcal{N}} z_k \Upsilon_k$, where Υ_k is short for Υ evaluated at t, s_k.

For fermions it is tempting to conjecture that

$$\Phi(t,..., s_k,..., s_l,...) - \Phi(t,..., s_l,..., s_k,...) = (2Z+1)\,\pi \tag{72}$$

where $Z \in \mathbb{Z}$. However, as James Taylor kindly pointed out to me, this possibility is spoiled through a subtle topological aspect of a Fermi bundle; we return to fermions in the sequel to this paper.

4.5. The Conservation Laws

Interestingly, a rigorous study of the conservation laws associated with the gauge invariance and the spacetime symmetries of the theory, i.e., the conservation of charge, energy, momentum, angular momentum, and moment of energy-momentum, shows that they hold essentially in the forms anticipated in ref. 13 by symbolic manipulations that pretend the validity of Newton's law of motion with a formal Lorentz force. The conservation laws reduce to the forms proposed by Born and Infeld[22] only in the absence of charges.

4.5.1. Fields with Point Charge Sources

In the presence of point sources the functionals of total charge, energy, momentum, angular momentum, and moment of energy-momentum take the following form.

The functional of the actual total electric charge is given by

$$\mathcal{Q}(\Omega^{\mathrm{cl}}) := \mathcal{Q}_{\mathrm{field}}(\boldsymbol{D}), \tag{73}$$

where

$$\mathcal{Q}_{\mathrm{field}}(\boldsymbol{D}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\cdot\boldsymbol{D}\,\mathrm{d}^3(s) \tag{74}$$

is the functional that counts the signed field defects.

The functional of the actual total energy is given by

$$\mathcal{E}(\Omega^{\mathrm{cl}}) = \mathcal{E}_{\mathrm{field}}(\boldsymbol{B},\boldsymbol{D}) + \sum_{k\in\mathcal{N}} \sqrt{1+|\nabla_k\Phi(t,\boldsymbol{R}) - z_k\alpha\boldsymbol{A}_k(t,\boldsymbol{R})|^2}, \tag{75}$$

where

$$\mathcal{E}_{\mathrm{field}}(\boldsymbol{B},\boldsymbol{D}) = \frac{1}{4\pi}\frac{\alpha}{\beta^4}\int_{\mathbb{R}^3}\left(\sqrt{1+\beta^4(|\boldsymbol{B}|^2+|\boldsymbol{D}|^2)+\beta^8\,|\boldsymbol{B}\times\boldsymbol{D}|^2}-1\right)\mathrm{d}^3(s) \tag{76}$$

is the functional of the field energy. At the same time, the actual field-energy functional is the field Hamiltonian for the conjugate field variables $\boldsymbol{B}$ and $\boldsymbol{D}$; to emphasize this we shall sometimes use the notation $\mathcal{H}_{\mathrm{field}}(\boldsymbol{B},\boldsymbol{D})$ instead of $\mathcal{E}_{\mathrm{field}}(\boldsymbol{B},\boldsymbol{D})$.

The functional of the actual total momentum of electromagnetic field plus defects reads

$$\mathcal{P}(\Omega^{\mathrm{cl}}) = \mathcal{P}_{\mathrm{field}}(\boldsymbol{B},\boldsymbol{D}) + \sum_{k\in\mathcal{N}} (\nabla_k\Phi(t,\boldsymbol{R}) - z_k\alpha\boldsymbol{A}_k(t,\boldsymbol{R})), \tag{77}$$

where

$$\mathscr{P}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) = \frac{\alpha}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{D}\times\boldsymbol{B}\,\mathrm{d}^3(s) \tag{78}$$

is the functional of the field momentum.

The functional of the actual total angular momentum is given by

$$\mathscr{J}(\Omega^{\text{cl}}) = \mathscr{J}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) + \sum_{k\in\mathscr{N}} \boldsymbol{r}_k \times (\nabla_k \Phi(t, \boldsymbol{R}) - z_k\alpha \boldsymbol{A}_k(t, \boldsymbol{R})), \tag{79}$$

where

$$\mathscr{J}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) = \frac{\alpha}{4\pi}\int_{\mathbb{R}^3} \boldsymbol{s}\times(\boldsymbol{D}\times\boldsymbol{B})\,\mathrm{d}^3(s), \tag{80}$$

is the functional of the field angular momentum.

The functional of the actual moment of the total energy is given by

$$\mathscr{M}(\Omega^{\text{cl}}) = \mathscr{M}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) + \sum_{k\in\mathscr{N}} \sqrt{1+|\nabla_k \Phi(t, \boldsymbol{R}) - z_k\alpha \boldsymbol{A}_k(t, \boldsymbol{R})|^2}\,\boldsymbol{r}_k, \tag{81}$$

where

$$\mathscr{M}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) = \frac{1}{4\pi}\frac{\alpha}{\beta^4}\int_{\mathbb{R}^3} \left(\sqrt{1+\beta^4(|\boldsymbol{B}|^2+|\boldsymbol{D}|^2)+\beta^8\,|\boldsymbol{B}\times\boldsymbol{D}|^2}-1\right)\boldsymbol{s}\,\mathrm{d}^3(s) \tag{82}$$

is the functional of the moment of the field energy.

The conservation laws in the presence of point sources are collected in

Proposition 4.2. For $t\in(0,T)$, let $t\mapsto\Omega^{\text{cl}}(t)\in\Gamma^{\text{cl}}$ be a regular solution of the Cauchy problem with point sources, satisfying $\lim_{t\downarrow 0}\Omega^{\text{cl}}(t) = \Omega^{\text{cl}}(0)$ for prescribed initial data $\Omega^{\text{cl}}(0)\in\Gamma^{\text{cl}}$. Then the conventional physical conservation laws for total charge, energy, momentum, and angular momentum are satisfied, i.e.,

$$\mathscr{Q}(\Omega^{\text{cl}}(t)) = Q, \tag{83}$$

$$\mathscr{E}(\Omega^{\text{cl}}(t)) = E, \tag{84}$$

$$\mathscr{P}(\Omega^{\text{cl}}(t)) = \boldsymbol{P}, \tag{85}$$

$$\mathscr{J}(\Omega^{\text{cl}}(t)) = \boldsymbol{J}, \tag{86}$$

with Q, E, $\boldsymbol{P}$, and $\boldsymbol{J}$ independent of time.

Furthermore, the moment of total energy and the total momentum are related by

$$\mathscr{M}(\Omega^{\mathrm{cl}}(t)) - t\mathscr{P}(\Omega^{\mathrm{cl}}(t)) = \boldsymbol{M}, \tag{87}$$

with $\boldsymbol{M} = \mathscr{M}(\Omega^{\mathrm{cl}}(0))$ independent of time.

Sketch of Proof of Proposition 4.2. The law of the charge conservation holds because the continuity equation for the point charge and current densities holds in lieu of their definitions, and since $\nabla \cdot \boldsymbol{D} = 4\pi j$ by Coulomb's law (29).

To prove the other conservation laws, regularize A and $\boldsymbol{A}$ by convolution with a differentiable compactly supported probability density having spherical symmetry in foliation space. The conservation laws for the regularized Hamilton–Jacobi dynamics can then be proved by differentiation in a similar straightforward fashion as done in ref. 62 for the old Abraham–Lorentz model. Since the limit as the regularizer concentrates on a point is absolute in the Hamilton–Jacobi formulation, not merely conditional as in the Newtonian formulation, the conservation laws for the point charge model follow. ∎

One can easily show that the conserved quantities total energy, total momentum, total angular momentum, and the moment of total energy-momentum are the generators of the time translations, space translations, space rotations, and spacetime boosts. The total charge generates the gauge transformations. As a general reference to symmetries and conservation laws, see ref. 98.

4.5.2. Source-Free Fields

In the absence of charges, when the evolution equations are reduced to the Maxwell–Born–Infeld field equations without source terms, the conservation laws of Proposition 4.2 reduce to their obvious corollaries. In addition, as is the case with the vacuum Maxwell equations, the source-free Maxwell–Born–Infeld field equations and their Hamiltonian $\mathscr{H}_{\mathrm{field}}$ are also invariant under electric-magnetic duality transformations (generalized Hodge duality rotations; "γ transformations" in ref. 91), which implies a conservation law for the sum of the electric and magnetic field helicities, cf. ref. 11. These are defined as

$$\mathscr{Y}_{\mathrm{field}}(\boldsymbol{B}) = \frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{B}(\hat{s}) \cdot \nabla \times \boldsymbol{B}(\check{s})}{|\hat{s} - \check{s}|}\, \mathrm{d}^3(\hat{s})\, \mathrm{d}^3(\check{s}) \tag{88}$$

and

$$\mathscr{Y}_{\text{field}}(\boldsymbol{D}) = \frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\boldsymbol{D}(\hat{s})\cdot\nabla\times\boldsymbol{D}(\check{s})}{|\hat{s}-\check{s}|}\,\mathrm{d}^3(\hat{s})\,\mathrm{d}^3(\check{s}). \tag{89}$$

We summarize the conservation laws for the source-free evolution.

Proposition 4.3. For $t\in(0,T)$, let the pair of maps $t\mapsto \boldsymbol{B}(t,.)$ and $t\mapsto \boldsymbol{D}(t,.)$ be a classical solution of the source-free Maxwell–Born–Infeld field equations, satisfying $\lim_{t\downarrow 0}\boldsymbol{B}(t,.)=\boldsymbol{B}(0,.)$ and $\lim_{t\downarrow 0}\boldsymbol{D}(t,.)=\boldsymbol{D}(0,.)$ for the prescribed initial data $\boldsymbol{B}(0,.)$ and $\boldsymbol{D}(0,.)$. Then the conventional physical conservation laws for the charge, field energy, field momentum, and field angular momentum are satisfied, i.e.,

$$\mathscr{Q}_{\text{field}}(\boldsymbol{D}(t,.)) = 0, \tag{90}$$

$$\mathscr{H}_{\text{field}}(\boldsymbol{B}(t,.),\boldsymbol{D}(t,.)) = E_{\text{field}}, \tag{91}$$

$$\mathscr{P}_{\text{field}}(\boldsymbol{B}(t,.),\boldsymbol{D}(t,.)) = \boldsymbol{P}_{\text{field}}, \tag{92}$$

$$\mathscr{J}_{\text{field}}(\boldsymbol{B}(t,.),\boldsymbol{D}(t,.)) = \boldsymbol{J}_{\text{field}}, \tag{93}$$

with E_{field}, $\boldsymbol{P}_{\text{field}}$, and $\boldsymbol{J}_{\text{field}}$ independent of time.

In addition, the sum of electric and magnetic field helicities is conserved, i.e.,

$$\mathscr{Y}_{\text{field}}(\boldsymbol{B}(t,.)) + \mathscr{Y}_{\text{field}}(\boldsymbol{D}(t,.)) = Y_{\text{field}}, \tag{94}$$

with Y_{field} independent of time.

Also, the moment of the field energy and the field momentum are related by

$$(\mathscr{M}_{\text{field}} - t\mathscr{P}_{\text{field}})(\boldsymbol{B}(t,.),\boldsymbol{D}(t,.)) = \boldsymbol{M}_{\text{field}}, \tag{95}$$

with $\boldsymbol{M}_{\text{field}}$ independent of time.

Sketch of proof of Proposition 4.3. Proposition 4.3 is largely a special case of our proof of Proposition 4.2, except for the proof of the conservation law for the electromagnetic field helicity, which can be done as yet another exercise in vector analysis.

An alternate proof, using the generators of the various symmetries associated with the conservation laws (except helicity), can be found in ref. 13, pp. 92–95; the corresponding proof of the conservation law for the electromagnetic field helicity (though it is not called by that name) can be found in ref. 11, pp. 41/42. ∎

We remark that in the two limiting cases $\beta \to 0$ and $\beta \to \infty$, both of which yield field equations that are invariant under the full conformal group on Minkowski space, further conservation laws of the source-free field dynamics emerge.[11] These additional conservation laws concern electric and magnetic field helicities separately, as well as an electromagnetic cross-field helicity,

$$\mathscr{X}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\boldsymbol{B}(\hat{\boldsymbol{s}}) \cdot \nabla \times \boldsymbol{D}(\check{\boldsymbol{s}})}{|\hat{\boldsymbol{s}} - \check{\boldsymbol{s}}|} \, \mathrm{d}^3(\hat{\boldsymbol{s}}) \, \mathrm{d}^3(\check{\boldsymbol{s}}). \tag{96}$$

5. SPECIAL TYPES OF SOLUTIONS OF THE MAXWELL–BORN–INFELD FIELD EQUATIONS WITH POINT SOURCES

We now list some physically relevant facts about special solutions of the electromagnetic Maxwell–Born–Infeld field equations with $\beta \in (0, \infty)$.

5.1. On the Existence and Uniqueness of Electrostatic Solutions

Everything that is rigorously known about solutions with point sources refers to solutions which are static in some Lorentz frame after at most a Lorentz boost. Therefore it suffices to discuss the static versions.

5.1.1. The Electric Field of N Unit Point Charges

For an electrostatic solution, $\boldsymbol{B} = \boldsymbol{0} = \boldsymbol{H}$, while the electrostatic field $\boldsymbol{s} \mapsto \boldsymbol{E}$ and the electric displacement field $\boldsymbol{s} \mapsto \boldsymbol{D}$ satisfy the electrostatic Maxwell equations

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{97}$$

and

$$\nabla \cdot \boldsymbol{D} = 4\pi \sum_{k \in \mathcal{N}} z_k \delta_{s_k} \tag{98}$$

in an aether governed by the electrostatic Born and Infeld law

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\sqrt{1 + \beta^4 \, |\boldsymbol{D}|^2}}. \tag{99}$$

Proposition 5.1. A unique finite-energy solution to the static Maxwell–Born–Infeld field Eqs. (97)–(99) with $N \in \mathbb{N}$ fixed unit point charges as sources exists whenever there exists a maximal space-like slice in Minkowski space with N null-like point defects at the locations of the charges.

Sketch of Proof of Proposition 5.1. To have finite energy $\mathscr{E}(\Omega^{\mathrm{cl}}_{\mathrm{static}}) = \hat{\mathscr{E}}(\boldsymbol{D})$,

$$\hat{\mathscr{E}}(\boldsymbol{D}) = N + \frac{1}{4\pi}\frac{\alpha}{\beta^4}\int_{\mathbb{R}^3}\left(\sqrt{1+\beta^4\,|\boldsymbol{D}|^2}-1\right)\mathrm{d}^3(\boldsymbol{s}), \tag{100}$$

we need to impose the asymptotic condition that $\boldsymbol{D}(\boldsymbol{s}) \to \boldsymbol{0}$ as $|\boldsymbol{s}| \to \infty$, which is inherited by $\boldsymbol{E}$. Then (97) is satisfied identically if there exists an electrostatic scalar potential $\boldsymbol{s} \mapsto A$, vanishing for $|\boldsymbol{s}| \to \infty$, such that

$$\boldsymbol{E}(\boldsymbol{s}) = -\nabla A(\boldsymbol{s}) \tag{101}$$

for all $\boldsymbol{s} \neq \boldsymbol{s}_k$, $k \in \mathscr{N}$. We next invert the electrostatic Born and Infeld law of the aether (99) to expresses $\boldsymbol{D}$ explicitly in terms of ∇A, viz.

$$\boldsymbol{D} = -\frac{\nabla A}{\sqrt{1-\beta^4\,|\nabla A|^2}}. \tag{102}$$

Taking now the divergence of (102) and noting (98) gives

$$-\nabla\cdot\frac{\nabla A}{\sqrt{1-\beta^4\,|\nabla A|^2}} = 4\pi\sum_{k\in\mathscr{N}} z_k\delta_{\boldsymbol{s}_k}, \tag{103}$$

together with the asymptotic condition that $A(\boldsymbol{s}) \to 0$ for $|\boldsymbol{s}| \to \infty$.

All the hard work has been absorbed in the single, nonlinear, second-order partial differential Eq. (103) for A. But (103) is the Euler–Lagrange equation for

$$\int_{\mathbb{R}^3}\left(\sqrt{1-\beta^4\,|\nabla A|^2}-1\right)\mathrm{d}^3(\boldsymbol{s}) + 4\pi\beta^4\sum_{k\in\mathscr{N}} z_k A(\boldsymbol{s}_k) = \text{maximum}, \tag{104}$$

for $A \in C^{0,1}_0(\mathbb{R}^3) \cap C^1(\mathbb{R}^3\setminus\{\boldsymbol{s}_k\}_{k=1}^N)$ with $\mathrm{Lip}(A) = \beta^{-2}$. With $\beta^2 A$ interpreted as the time function T of the Section 3.1.4, with lapse function $\ell = 1$, see (6), the variational principle (104) characterizes a maximal space-like slice in Minkowski space with N null-like point defects at the locations of the charges, and (103) states that the mean curvature of that space-like slice vanishes except at the defects. This concludes the existence part of the proposition. We remark that theorem 5.4, and Remark 1 thereafter, of ref. 7 seems to guarantee the existence of the solutions of (104); see also refs. 64 and 65 for results in certain domains with boundary.

As for the uniqueness of an electrostatic solution, it suffices to discuss uniqueness of an asymptotically (for $|s| \to \infty$) vanishing solution A of (103) for any given configuration of the N charges. The argument is based on convexity, entirely standard, and a little shorter than Pryce's proof.[86] Thus assume that two distinct solutions $A_1 \not\equiv A_0$ of (103) exist for a given configuration of the N charges. We subtract (103) for A_0 from (103) for A_1, multiply by $A_1 - A_0$, integrate over $\mathbb{R}^3$, use integration by parts and obtain the identity

$$0 = \int_{\mathbb{R}^3} \nabla(A_1 - A_0) \cdot \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u} \frac{\nabla A_u}{\sqrt{1 - \beta^4 \, |\nabla A_u|^2}} \, \mathrm{d}u \, \mathrm{d}^3(s) \tag{105}$$

where $A_u = uA_1 + (1-u)\, A_0$. Now exchange the volume and the u integrations, carry out the indicated u-differentiation of the integrand, and obtain a manifestly positive definite integral on the r.h.s. which vanishes iff $A_1 - A_0 \equiv 0$. ▮

5.2. Special Electrostatic Solutions

5.2.1. Born's Solution (The Electric Field of a Single Unit Point Charge)

The static Maxwell–Born–Infeld equations with a single point charge as source can be solved explicitly, as already announced in ref. 16 and elaborated on further in refs. 17, 20, and 21. We identify the origin of space with the location of the charge at rest. The electric displacement field of Born's solution is identical to the Coulomb field of a point charge,

$$\boldsymbol{D}^{\pm}_{\mathrm{Born}}(\boldsymbol{s}) = \boldsymbol{D}^{\pm}_{\mathrm{Coulomb}}(\boldsymbol{s}) \equiv \pm \frac{\boldsymbol{s}}{|\boldsymbol{s}|^3}, \tag{106}$$

where $\pm$ indicates a positive or negative electron. The associated electric field $\boldsymbol{E}^{\pm}_{\mathrm{Born}}$ is bounded, but undefined at the origin. For $\boldsymbol{s} \neq \boldsymbol{0}$, $\boldsymbol{E}^{\pm}_{\mathrm{Born}}(\boldsymbol{s}) = -\nabla A^{\pm}_{\mathrm{Born}}(\boldsymbol{s})$, where

$$A^{\pm}_{\mathrm{Born}}(\boldsymbol{s}) = \pm \frac{1}{\beta} \int_{|\boldsymbol{s}|/\beta}^{\infty} \frac{\mathrm{d}x}{\sqrt{1 + x^4}}. \tag{107}$$

This spherically symmetric electrostatic potential is asymptotic to Coulomb's potential, $A^{\pm}_{\mathrm{Born}}(\boldsymbol{s}) \sim \pm |\boldsymbol{s}|^{-1}$ for $|\boldsymbol{s}| \gg \beta$, and $\lim_{|\boldsymbol{s}| \to 0} A^{\pm}_{\mathrm{Born}}(\boldsymbol{s}) = A^{\pm}_{\mathrm{Born}}(\boldsymbol{0}) < \infty$.

In ref. 86 the uniqueness of (107) under the condition of finite field energy is shown, by an argument similar in spirit to the one we used above.

Ecker[41] showed that, interpreted as the time function of a maximal space-like slice with defect, Born's solution is the unique solution to (104) among all asymptotically flat space-like slices with any single isolated singularity; note that the singularity of Born's solution is a so-called (light)cone singularity in the space-like slice interpretation.

5.2.2. Hoppe's Solution (The Electric Field of an Infinite Crystal of Unit Point Charges)

If we allow $N \to \infty$ and relax the condition of finite energy, another exact many-body solution of (103) for $\beta \in (0, \infty)$ becomes available, which was discovered by Hoppe.[52, 53] It is defined implicitly by

$$\wp(A_{\text{Hoppe}}(s^1, s^2, s^3)) = \wp(s^1)\,\wp(s^2)\,\wp(s^3) \tag{108}$$

where $\wp$ is the elliptic Weierstrass function.[33] This solution describes an infinite "electron-positron crystal" with NaCl symmetry. For more, see the review ref. 45.

6. ILLUSTRATIONS OF THE ELECTROMAGNETIC CAUCHY PROBLEM WITH POINT CHARGES

We finally have collected all the relevant pieces of information that we need for explicitly illustrating the setup of the electromagnetic Cauchy problem for fields and point charges. It suffices to consider fields coupled to a single electron.

6.1. A Single Electron at Rest Surrounded by Its Own Electrostatic Field

We begin with the simplest possible case: an electron initially at rest somewhere in space, surrounded only by its own electrostatic field. Even this simplest one-particle case is already too much to handle for the Newtonian law of motion with the total Lorentz force, while for our Hamilton–Jacobi law of motion it is trivial.

Clearly, the initial $\boldsymbol{A}$ vanishes identically, so we can take the initial $\boldsymbol{A}^\sharp$ to vanish identically too. Also, the initial $\boldsymbol{D}$ is just the Coulomb field of the point charge, with the corresponding electrostatic potential field in space

[33] Recall that $\zeta \mapsto \wp(\zeta)$ satisfies the nonlinear ODE $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ with invariants $g_2 = 60\sum_{(n,m)\in\mathbb{Z}^2_*}(n2\omega + m2\omega')^{-4}$ and $g_3 = 140\sum_{(n,m)\in\mathbb{Z}^2_*}(n2\omega + m2\omega')^{-6}$, where $\mathbb{Z}^2_* = \mathbb{Z}^2\backslash(0,0)$, and ω and ω' are the real half-periods (note the bizarre notation is standard). $\wp$ has a pole of order two at the origin; more precisely, $\wp(\zeta) - \zeta^{-2}$ is analytic in a neighborhood of the origin.

given by Born's solution (107) if the electron is at the origin of space, and by a translate of it if the electron is elsewhere. This fixes the initial $\boldsymbol{D}^\sharp$ and $A^\sharp$; the latter is the corresponding translate in space by s_1 of (107). Lastly, since the particle is initially at rest no matter where it is, we take $\boldsymbol{V}(0, s_1) \equiv \boldsymbol{0}$, and $\Phi(0, s_1) \equiv 0$.

It is now easy to see that these data imply that $A^\sharp(0, s_1, s_1) = A^{(-)}_{\text{Born}}(\boldsymbol{0})$ for all s_1. Therefore the Hamilton–Jacobi PDE for Φ initially reads $\partial\Phi = K$, a constant function in $\mathbb{R}^3_1$, while $\partial \boldsymbol{X}^\sharp = \boldsymbol{0}$ for all $^\sharp$fields $\boldsymbol{X}^\sharp$. Hence, the $^\sharp$fields initial data form a stationary solution of the $^\sharp$field equations, while the Hamilton–Jacobi equation is solved by $\Phi(t, s_1) = Kt$. This gives $\boldsymbol{V}(t, s_1) \equiv \boldsymbol{0}$ for all time, which implies that the actual particle position and, hence, that the actual electromagnetic fields and their potentials on space, all retain their initial data for all time, as they should.

We take the opportunity to re-enforce what we emphasized earlier in Section 3, namely that Φ should not be interpreted as a field on physical space. Indeed, if instead of for a field Φ on configuration space we had misinterpreted the Hamilton–Jacobi PDE as a scalar equation for a field ϕ on physical space, say with the particle initially at the origin, then initially and near the origin of space that field equation would read qualitatively like $\partial\phi = b - |s|$. Now ϕ too would immediately develop a kink, as function of space, at the location of the point charge, and the guiding equation would become as meaningless as the Newtonian equation of motion. The same remark applies verbatim to the upgraded test particles Hamilton–Jacobi theory.

6.2. A Single Electron and an Oppositely Charged, Infinitely Massive Nucleus, Both Initially at Rest in the Total Electrostatic Field

To have another, less trivial example, consider a single negative electron initially at rest at $\boldsymbol{r}_1(0) = \boldsymbol{r}_0$ and an infinitely massive positive unit point charge at rest at the origin of space, so that we are allowed to restrict our discussion of configuration space to the single electron space. The initial $\Phi(0, s_1)$ on the single electron configuration space is a constant, Φ_0 say. The initial electromagnetic fields are electrostatic, the field generated by the two point charges, for which existence we once again invoke Bartnik's results (see above). The time derivative of these initial fields in space vanishes initially. Again, we take the initial magnetic potential to vanish identically, and do the same for its magnetic $^\sharp$cousin. Unfortunately, to set up the initial value problem for the $^\sharp$fields *explicitly*, we need to compute the initial total electrostatic displacement field and potential on space-configuration space, $\boldsymbol{D}^\sharp(0, s, s_1)$ and $A^\sharp(0, s, s_1)$, for which we need to solve the electrostatic Maxwell–Born–Infeld equations with two-point sources, and which no one has been able to do, as far as we know. However, to set up the initial value problem for $\Phi(t, s_1)$, we only need to know the initial $A_1(0, s_1) = A^\sharp(0, s_1, s_1)$, for which we were able to find an explicit integral representation. Clearly, $A_1(0, s_1)$ is here a nontrivial function of $|s_1|$. More precisely, $A_1(0, s_1)$ is real analytic as function of $|s_1| \in (0, \infty)$, (extended to $|s_1| \in \mathbb{R}$) its Taylor series about the origin has finite radius of convergence and begins as

$$A_1(0, s_1) = -\frac{1}{2\beta}\left[\frac{|s_1|}{\beta} - O\left(\frac{|s_1|^5}{\beta^5}\right)\right], \tag{109}$$

and the asymptotic expansion for large $|s_1|$ reads

$$A_1(0, s_1) = A_{\text{Born}}^{(-)}(\boldsymbol{0}) + \frac{1}{|s_1|}\left[1 - U\left(\frac{\beta}{|s_1|}\right)\right], \tag{110}$$

with $|U(\beta/|s_1|)| < C\beta/|s_1|$ for large $|s_1|$. This result is just part of a more detailed theorem which we need for our discussion of the hydrogen spectrum, and for which reason we defer its rigorous proof to our follow-up paper on the partially quantized theory.[63] Here we just note that our results show that the Hamilton–Jacobi PDE is well-defined initially with these data $A_1(0, s_1)$ and Φ_0. Different from the first, trivial example, the Hamilton–Jacobi PDE now launches a solution $\Phi(t, s_1)$ which immediately develops a spatial dependence on the position in configuration space. Its non-trivial configuration space gradient (viz., velocity field on configuration space) is well-defined everywhere except at the origin, where $\Phi(t, s_1)$ develops a kink. Thus, since the point electron cannot be initially at rest exactly on top of the nuclear point charge, for coincidence points are removed from configuration space, in its immediate future it begins to move according to its guiding equation. Better than that, our asymptotic expansion of $A_1(0, s_1)$ also shows that to leading order after the irrelevant constant term $A_{\text{Born}}^{(-)}(\boldsymbol{0})$ the negative point electron sees the familiar electrostatic Coulomb potential of the infinitely massive positive point charge at the origin. Thus, not only does the point electron begin to move; if it is initially far enough from the origin, then in leading order of the asymptotic expansion it begins to move precisely according to the familiar Newtonian law of motion in a *given* attractive Coulomb field, as it should.

As to the question of global well-posedness vs. finite-time blow-up, we remark that it is to be expected that the evolution of the point electron will run into the nuclear point charge in finite time, which is the shorter the closer to the nucleus the point electron started out of rest initially. Once it has hit the nucleus, the motion of the point electron cannot be continued uniquely beyond this dynamical singularity, but until this happens we expect the evolution to be regular. In any event, we have not tried to prove this yet.

We also remark that the initial β-correction is tiny unless the two charges are sufficiently close together. How close is "sufficiently close" will be of relevance for our next section, the assessment of Born's calculation of β.

7. THE VALUES OF THE UNIVERSAL CONSTANTS α AND β (PRELIMINARY ASSESSMENT)

We are finally ready to vindicate our identification of the universal constant α in our guiding laws with Sommerfeld's fine structure constant, as indeed done in (1). However, we have collected barely enough material to identify β correctly. We begin with the tentative determination of α/β.

7.1. Born's Determination of α/β

Inspired by the idea of the later 19th century that the electron's inertia a.k.a. mass has a purely electromagnetic origin, Born argued that the electrostatic energy of the spherically symmetric Born alias Coulomb field $\boldsymbol{D}_{\text{Born}}$ of the electronic point charge at rest be identified with the empirical rest energy of the electron.(16) Since no other fields are present than the Coulomb field, we have $\boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{D} = \boldsymbol{D}^{\pm}_{\text{Born}}$, where the superscript $\pm$ again indicates the sign of the electron's charge. The electrostatic field energy of Born's solution is therefore given by

$$\mathscr{H}_{\text{field}}(\boldsymbol{0}, \boldsymbol{D}^{\pm}_{\text{Born}}) = \frac{1}{4\pi}\frac{\alpha}{\beta^4}\int_{\mathbb{R}^3} \left(\sqrt{1+\beta^4\,|\boldsymbol{D}^{\pm}_{\text{Born}}|^2} - 1\right) \mathrm{d}^3(\boldsymbol{s}), \tag{111}$$

which evaluates independently of the sign of the charge to

$$\mathscr{H}_{\text{field}}(\boldsymbol{0}, \boldsymbol{D}^{\pm}_{\text{Born}}) = \frac{\alpha}{\beta}\frac{1}{6}\,\mathrm{B}\left(\frac{1}{4}, \frac{1}{4}\right), \tag{112}$$

where $\mathrm{B}(p, q)$ is Euler's Beta function. In our dimensionless formulation, the empirical rest energy $m_{\mathrm{e}}c^2$ of the electron is the unit of energy. Hence, setting

$$\mathscr{H}_{\text{field}}(\boldsymbol{0}, \boldsymbol{D}^{\pm}_{\text{Born}}) = 1, \tag{113}$$

we find Born's result[(16)][34]

$$\left.\frac{\beta}{\alpha}\right|_{\text{Born}} = \frac{1}{6}\,\mathrm{B}\left(\frac{1}{4}, \frac{1}{4}\right) \approx 1.2361. \tag{114}$$

Remark that with Born's value for β, the value at the origin of the electrostatic potential of Born's solution (107) is given by

$$\alpha A^{\pm}_{\text{Born}}(\boldsymbol{0}) = \pm\tfrac{3}{2}. \tag{115}$$

Comments on Born's Calculation of α/β

The derivation of formula (114) is not as unproblematic as it pretends to be.

Born's thoughts about the purely electromagnetic origin of the electron's inertia were reinforced by his and Infeld's conviction that the field energy functional $\mathscr{H}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D})$ was the conserved total energy quantity, which our energy conservation theorem shows not to be true in the presence of point charges. Moreover, the qualitative content of the law of energy conservation does not change if an arbitrary constant is added to the expression on the r.h.s. of (75), but clearly its quantitative content does. One might want to argue on behalf of the mathematical integrity of (75) that the adding of any nontrivial constant, other than perhaps -1 per single electron,[35] would be totally perverse unless compelling reasons for such an additive constant would be found elsewhere, but this is not an entirely convincing way of reasoning. In any event, Born's calculation of the value of α/β suddenly seems rather arbitrary. An entirely unambiguous identification of α/β or even β can only be made on the basis of truly dynamical considerations.

[34] As mentioned before, Born used a different, dimensional notation; our $\beta^2 \propto a$ of Born.

[35] As for adding -1 per electron to the r.h.s. of (75), this gives the alternate total energy functional

$$\tilde{\mathscr{E}}(\Omega^{\text{cl}}) = \mathscr{H}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) + \sum_{k\in\mathcal{N}} \left(\sqrt{1 + |\nabla_k \Phi(t, \mathbf{R}) - z_k \alpha \boldsymbol{A}_k(t, \mathbf{R})|^2} - 1\right). \tag{116}$$

For a system containing only a single electron at rest, the energy functional $\tilde{\mathscr{E}}(\Omega^{\text{cl}}_0)$ coincides with the field-energy functional $\mathscr{H}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D})$. The identification of the electron's empirical rest energy with either the electrostatic field energy or the total energy of the static single-electron state then gives the same result, which would seem like an attractive way out of the dilemma.

The first clues can be obtained by taking the other conservation laws into account, which already provide some additional pieces of information of the underlying dynamical system. In particular, the usually-little-attention-paid-to law of the linear motion of the moment of energy (87) suggests that the total energy as identified in (75) is singled out among all other possibilities that differ from (75) by an additive constant, whether per electron or in total. Once we have accepted (75) as the likely candidate for the correct total energy, the next question is whether it is still a reasonable working hypothesis to identify the electron's rest energy ($=1$) with the electrostatic field energy, or whether we should identify it with the total energy of the static single-electron state, or neither of the two.

With the help of elementary considerations we can immediately dispose of the option of identifying the total energy (75) of the electrostatic single-electron state with the electron's rest energy ($=1$), for this leads to the conclusion $\alpha\beta^{-4}=0$. However, $\alpha=0$ means the particles feel no influence of the electromagnetic fields, so we need to have $\alpha\neq 0$. The only other possibility, then, is to let $\beta\to\infty$, which gives us the ultra Born–Infeld laws (34) and (35). However, the ultra Born–Infeld field equations allow electrostatic solutions with arbitrarily many point charges placed arbitrarily in space, and the total energy of any such static N-electrons state is then always simply N, which is clearly incorrect. Hence, the alternate identification of the electron rest energy with the total energy (75) of the static single-electron state is not feasible.

Incidentally, we have actually disposed of either of the possibilities $\alpha=0$ and $\beta=\infty$. As we will see next, this in turn suffices to determine α, and from there to get some upper estimate on β, which comes close to the value given by Born's result. While this gives some a-posteriori confidence in the viability of Born's formula (114), we will re-assess the β value in our follow-up paper[(63)] on the spinless quantum theory, and once again when spin and photon are incorporated into the theory.

7.2. Identification of α with Sommerfeld's Fine Structure Constant

We now vindicate our identification $\alpha=e^2/\hbar c$ by showing formally that in the limit of radiation-reaction-free gentle motions, our guiding equation for a single electron reduces to the correct law of motion. Incidentally, the second example of our section illustrating the Cauchy problem already vindicates our claim about α for the special two-body Coulomb problem with one particle infinitely massive. We here adapt this line of reasoning to the more general situation depicted below. Another, more rigorous vindication will be supplied in our follow-up paper[(63)] on the quantum theory.

We consider a Lorentz frame in which a single (negative) point electron is deflected (in Born–Oppenheimer approximation) in the Coulomb field $\boldsymbol{E}_{\text{Coulomb}}$ of an infinitely massive point charge at rest, and perhaps also by an electromagnetic radiation field with very low intensity and very long wavelength which impinges on the electron. For the notion of Coulomb field $\boldsymbol{E}_{\text{Coulomb}}$ to apply, we need to have $|\boldsymbol{s}| \gg \beta$, as a perusal of (107) reveals; similarly, "long wavelength λ" is defined as $\lambda \gg \beta$, while "low intensity" is defined as $\beta^2(|\boldsymbol{B}|+|\boldsymbol{D}|) \ll 1$. These definitions of smallness make sense as long as $\beta \in (0, \infty)$. Recall that $\beta = 0$ yields just the ill-defined Lorentz electrodynamics with point charges, while $\beta = \infty$ yields the fully conformally invariant ultra Maxwell–Born–Infeld field equations with point charge sources, which we have just disposed of; hence, we do have $\beta \in (0, \infty)$ and can proceed unimpeded.

When a particle moves at high speeds, but is only gently accelerated by the radiation field and the static Coulomb field, then according to the established physics, the asymptotically correct evolutionary law of the point electron is Newton's law of radiation-reaction-free motion, which equates the rate of change of kinematical particle momentum to a Lorentz force in which only the external (i.e., incoming radiation and static Coulomb) fields enter, while the kinematical particle momentum and particle velocity are related by Einstein's relativistic formula. It is an easy exercise to verify that the familiar textbook formulas, when converted from conventional dimensional Gaussian units into our dimensionless units, become

$$\dot{\boldsymbol{r}}(t) = \frac{\boldsymbol{p}(t)}{\sqrt{1+|\boldsymbol{p}(t))|^2}} \tag{117}$$

$$\dot{\boldsymbol{p}}(t) = -\frac{e^2}{\hbar c}\left(\boldsymbol{E}^{\text{ext}}(t, \boldsymbol{r}(t)) + \dot{\boldsymbol{r}}(t) \times \boldsymbol{B}^{\text{ext}}(t, \boldsymbol{r}(t))\right), \tag{118}$$

in which the dimensionless Sommerfeld fine structure constant $e^2/\hbar c$ features prominently as the only universal physical constant. We next argue, non-rigorously, that in the approximation in which radiation-reaction and non-linear field response is neglected, our Hamilton–Jacobi guiding laws reduce precisely to (117) and (118).

We define the external electric and magnetic fields $\boldsymbol{E}^{\text{ext}}(t, \boldsymbol{s})$ and $\boldsymbol{B}^{\text{ext}}(t, \boldsymbol{s})$ as solutions to the Maxwell–Born–Infeld field equations in $\mathbb{R}_+ \times \mathbb{R}^3$ with just the infinitely massive point source present. By solving the familiar partial differential Eqs. (36), (37), and (38) for appropriate initial conditions, and assuming vanishing conditions at spatial infinity, we obtain electric and magnetic potentials $A^{\text{ext}}(t, \boldsymbol{s})$ and $\boldsymbol{A}^{\text{ext}}(t, \boldsymbol{s})$ in the

Lorentz–Lorenz gauge which generate the external fields $\boldsymbol{E}^{\text{ext}}(t, \boldsymbol{s})$ and $\boldsymbol{B}^{\text{ext}}(t, \boldsymbol{s})$. Of course, the Lorentz–Lorenz gauge is also used for the electric and magnetic potentials of the total electric and magnetic fields, assuming the total potentials vanish at spatial infinity as well. On configuration space, we write $A_1 = A_1' + A^{\text{ext}}$ and $A_1 = A_1' + A^{\text{ext}}$. In the radiation-reaction-free approximation, and neglecting also the nonlinear modifications of the weak external fields at the location of the electron, we have (in 1-form notation, for brevity),

$$\mathbf{A}_1' \approx A_{\text{Born}}^{(-)}(\mathit{0})\,\mathbf{u} \tag{119}$$

with

$$A_{\text{Born}}^{(-)}(\mathit{0}) = -\frac{1}{4}\,\mathrm{B}\left(\frac{1}{4}, \frac{1}{4}\right)\beta^{-1}. \tag{120}$$

We may assume that in a local spacetime neighborhood of short truncated point histories, different initial $\mathbf{u}_0$ will define a vector field on configurational spacetime. In this neighborhood we then gauge-transform this convected electromagnetic potential away with the help of the gauge potential (0-form) $\Upsilon(t, \boldsymbol{s}) = -A_{\text{Born}}^{(-)}(\mathit{0}) \int_{\mathrm{H}_1} \mathbf{u}$, understood as a function of the endpoint of the integration along the truncated history. Thus, for such small times and regions in configuration space, after the radiation-reaction-free approximation (119) has been made, our Hamilton–Jacobi partial differential equation

$$\partial\Phi(t, s_1) = -\sqrt{1 + |\nabla_1\Phi(t, s_1) + \alpha(A_1' + A_1^{\text{ext}})(t, s_1)|^2} + \alpha(A_1' + A^{\text{ext}})(t, s_1) \tag{121}$$

and the Hamilton–Jacobi guiding equation

$$\frac{\mathrm{d}s_1}{\mathrm{d}t} = \frac{\nabla_1\Phi(t, s_1) + \alpha(A_1' + A^{\text{ext}})(t, s_1)}{\sqrt{1 + |\nabla_1\Phi(t, s_1) + \alpha(A_1' + A^{\text{ext}})(t, s_1)|^2}}, \tag{122}$$

are gauge equivalent to the Hamilton–Jacobi equations

$$\partial\Phi(t, s_1) = -\sqrt{1 + |\nabla_1\Phi(t, s_1) + \alpha A^{\text{ext}}(t, s_1)|^2} + \alpha A^{\text{ext}}(t, s_1), \tag{123}$$

and

$$\frac{\mathrm{d}s_1}{\mathrm{d}t} = \frac{\nabla_1\Phi(t, s_1) + \alpha A^{\text{ext}}(t, s_1)}{\sqrt{1 + |\nabla_1\Phi(t, s_1) + \alpha A^{\text{ext}}(t, s_1)|^2}}. \tag{124}$$

These equations are just the test particle Hamilton–Jacobi equations, well-known to be equivalent to (117) and (118), with α in place of $e^2/\hbar c$, which was to be shown.

7.3. An Upper Estimate for β

As the discussion in the previous section shows, only the information that $\beta \in (0, \infty)$, but not the precise value of the parameter β, enters in the derivation of the classical law of radiation-reaction-free motion of a point charge. It is clear that β will eventually enter in some higher-order-of-α corrections to this leading order law, but some early investigations by Schrödinger[(93)] indicate that to first order in radiation-reaction correction[36] the value of β still does not play any rôle. While Schrödinger's calculations are not based on a truly consistent dynamical model, we may have some confidence in his result because of the known fact that the first radiation-reaction correction to the equations of radiation-reaction-free motion (117), (118) is independent of any specific assumptions about the charge structure of the electron when computed from the classical electron theory,[(1, 74, 75, 96)] and also independent of the regularization in Dirac's re-calculation of this term for point electrons.[(36)] Although not a substitute for a rigorous discussion of radiation-reaction, this indicates that to see β enter explicitly in the asymptotic expansion of the equation of motion may require going to such high order in α that quantum physical corrections may be of equal importance. An assessment based on the hydrogen spectrum will be supplied in our next paper.

Yet, some quantitative information on β becomes available by rule of thumb if we ask for the empirical range of validity of the Eqs. (117), (118), which is quite impressive. Since for the derivation of (117), (118), we had to assume a low intensity and long wavelength of the incoming radiation, and a sufficiently large impact parameter for the scattering at the Coulomb potential, all of which terms are defined against β, the range of validity of (117), (118) sets some rough upper bound on β. In particular, from Coulomb scattering experiments with electrons we may have confidence in (117), (118) for length scales roughly down to the classical electron radius, which is a factor α smaller than the Compton wave length of the electron, our reference unit of length. This suggests an upper estimate on β roughly equal to α, so that the β value found by Born remains viable, for now.

We end by noting that the possibility $\beta = \alpha$ is currently as viable as Born's proposal (2), and so is Euler–Kockel's early QED result

[36] In our dimensionless units, this is an order α^2 correction to the r.h.s. of (118).

$\beta^4 = (\sharp/45\pi)\alpha^3$, with $\sharp \in [4, 7]$, giving $\beta/\alpha \approx 1.4$–1.6. However, future estimates based on the quantum theory may well rule out either of these possibilities.

8. SUMMARY AND OUTLOOK

In this paper we achieved the first consistent implementation of the notion of the point charge into the classical theory of electromagnetism. No regularization and no renormalization is called for. The actual electromagnetic fields are solutions of the Maxwell–Born–Infeld field equations with point charges sources, which move according to a relativistic guiding law of Hamilton–Jacobi type. The guiding field is generated by the solution of a relativistic Hamilton–Jacobi partial differential equation which is coupled self-consistently to the continuous potentials of generalized electromagnetic fields that live on space $\times$ configuration space. When the actual configuration is substituted for the generic one, then these generalized fields reduce to the actual electromagnetic fields on actual space. The formalism works because in the Maxwell–Born–Infeld theory of the electromagnetic fields, singularities associated with the point charges feature merely as mild defects in the electromagnetic potentials. It will not work for the fields of the older Maxwell–Lorentz theory with point charges. Curiously, since the (total) electromagnetic Maxwell–Born–Infeld fields are ill-defined at the actual positions of the point charges, the various generalized fields on space $\times$ configuration space that enter the Hamilton–Jacobi formalism cannot be eliminated, and this compels us to regard those fields not as mere mathematical auxiliary constructs, but as fields that enjoy a certain physical status of their own. This introduces a new element of physics into classical electrodynamics which is akin to the wave function in quantum physics.

Most of the paper is concerned with the consistent dynamical coupling of point charges to the classical electromagnetic Maxwell–Born–Infeld field equations, in an appendix we also address the question whether charge-free solitons with finite energy exist. There we prove rigorously that such soliton solutions of the Maxwell–Born–Infeld field equations cannot exist if their proper electric and magnetic field strengths remain below a huge threshold. Put differently, if such solitons exist, their peak field strengths must be enormous.

While in this paper we have worked out in detail only the special-relativistic theory, in an appendix on the action principle we note that the extension to a general-relativistic electromagnetic theory with point charges is feasible in which spacetime is not flat and static but curved and dynamical, and in which the electromagnetic stress-energy-momentum couples gravitationally to the spacetime as a source of its curvature. Interestingly enough, we anticipate that the point charges will appear as naked singularities of the spacetime; yet our formulation also yields a general relativistic Hamilton–Jacobi guiding law of motion for these singularities, thus indicating that the theory will contain enough information to remove the evolutionary ambiguities associated with the occurrence of naked singularities in general relativity. Technically, the nonlinear mathematical structure of Einstein's field equations poses formidable challenges to the rigorous implementation of this picture, but these will be taken up. It might be helpful to treat singularities as defects in the smoothness of spacetime, similar in spirit to the work[50] on the Schwarzschild metric, rather than punctures of spacetime. We plan to return to these issues at a later time.

Our special-relativistic theory should describe the physics of positive and negative point electrons and their radiation fields in the classical regime where spin effects and the photonic nature of the electromagnetic fields can be neglected. In this spirit, we have given the first assessment of the correctness of Born's value for his aether constant β, which enters the theory through the Born–Infeld laws of the aether. Born calculated the value of β by arguing that the empirical electron rest mass m_e $(\times c^2)$ be identified with the electrostatic energy of his spherically symmetric electrostatic solution for a single point charge. We showed that Born's argument, based as it is on his dynamically incomplete formulation of the theory, is inconclusive; yet Born's value for β remains viable for now, in the sense that it does not seem to conflict with any established classical electromagnetic effects. In particular, we showed for the two simplest examples of the initial value problem for a point charge that in leading order the conventional physical wisdom is reproduced. Our investigation led us to conclude that the β-induced corrections to these known effects in the classical domain of electromagnetism are presumably so small that they are masked by quantum effects. In other words, the definitive calculation of β can presumably be done only after the full quantization of our theory. A partial quantization of our theory, without spin and the photon, will be presented in ref. 63, the follow-up paper to this one. The incorporation of spin and the photon will be taken up subsequently. We add that, even though spin has not been incorporated at this classical level, our formalism has allowed us to implement the Pauli principle for many "bosonic electrons." We also add that only minor modifications of the theory are needed to accommodate the electromagnetic effects of other, non-genuinely electromagnetic particles, representing nuclei with or without magnetic moment and form factor. This requires putting in by hand the parameters z for the charge number, κ for the ratio of the electron's to the nucleus' rest mass, and, if desired, a smeared-out spinning charge distribution.

APPENDIX A

A.1. On Source-Free Solutions of the Maxwell–Born–Infeld Field Equations

In this appendix we collect some interesting results about solutions of the Maxwell–Born–Infeld field equations without sources (in particular without point sources) which, while pertinent to the content of the present paper, are somewhat beside its main thrust. We also prove a new no-soliton result.

All the known source-free solutions are electromagnetic waves of some sort.

A.1.1. Monochromatic Plane Waves

As observed already by Born (ref. 17, p. 434) and Schrödinger (ref. 91, p. 474), any monochromatic plane wave solution of Maxwell's field equations in vacuum also solves the source-free Maxwell–Born–Infeld field equations.[37] Indeed, in Maxwell's theory the electromagnetic vacuum fields satisfy $\boldsymbol{E}=\boldsymbol{D}$ and $\boldsymbol{B}=\boldsymbol{H}$, and for a monochromatic plane electromagnetic wave they satisfy also $|\boldsymbol{E}|=|\boldsymbol{B}|$ and $\boldsymbol{E}\cdot\boldsymbol{B}=0$. It is now an easy exercise to show that such field solutions of the vacuum Maxwell equations also satisfy the electromagnetic aether laws of Born and Infeld.[38]

The plane wave solutions themselves do not have a finite total energy even when their energy density is locally integrable. However, suitably cut off, plane wave solutions may provide useful approximations to solutions with finite total energy, and indeed are frequently used for this purpose.

[37] Curiously, Born and Infeld, apparently overlooking the fact that the electromagnetic field strengths in a plane monochromatic wave can be arbitrarily large, interpreted Born's field parameter b ($\propto \beta^{-2}$) (originally Born used $a \equiv b^{-1}$) as an upper bound to the field strengths by alluding to some "principle of finiteness" that Nature supposedly adheres to (p. 427 in refs. 21). It seems to have been Schrödinger who first pointed out that "none of the field quantities has an insurmountable upper limit in this theory" (p. 82 in ref. 92). This simple fact to the contrary notwithstanding, b continues to be misinterpreted as an absolute upper bound to the electric and magnetic field strengths even as recently as 2002, when some "extended relativity" was proposed in which, beside c as an absolute speed limit, eb/m_e is interpreted as absolute bound on accelerations.

[38] Schrödinger subsequently extended this result to a whole class of nonlinear electromagnetic field equations, see App.I in ref. 94.

A.1.2. Polychromatic Plane Waves with Linear Polarization

Unlike in Maxwell's linear electromagnetic vacuum theory, arbitrary linear superpositions of monochromatic plane waves will in general not furnish a solution of the Maxwell–Born–Infeld field equations, but certain linearly polarized polychromatic plane waves will do. More specifically, by at most an $SO(3)$ rotation of our coordinate system, we may assume that the plane wave propagates in the $\boldsymbol{e}_z$ direction, that $\boldsymbol{B}$ points along (or against) the $\boldsymbol{e}_x$ direction, while $\boldsymbol{E}$ points along (or against) the $\boldsymbol{e}_y$ direction. Let $p\colon \mathbb{R} \to \mathbb{R}$ by any differentiable function. Then

$$\boldsymbol{B}^{\pm}(t,z) = p(z \pm t)\, \boldsymbol{e}_x = \boldsymbol{H}^{\pm}(t,z), \tag{125}$$

$$\boldsymbol{D}^{\pm}(t,z) = -p(z \pm t)\, \boldsymbol{e}_y = \boldsymbol{E}^{\pm}(t,z) \tag{126}$$

with either sign solves the source-free Maxwell–Born–Infeld field equations. In particular, one can choose p to have pulse shape.

While the superposition of one left- and one right-propagating fixed pulse shape will in general not solve the Maxwell–Born–Infeld equations, this can provide interesting asymptotic conditions for studies of the nonlinear pulse interactions. Such studies have been carried out in refs. 46 and 54; see also the earlier review by Gibbons.(45) Two such pulses resemble two colliding solitons.[39]

A.1.3. Bichromatic Plane Waves with Circular Polarizations

A bichromatic plane wave solution with circular polarizations was discovered by Schrödinger,(94) see refs. 45 for a recent discussion. Schrödinger's solution was perhaps the first hint at some hidden continuous symmetries of the source-free Maxwell–Born–Infeld equations that may lead to further conservation laws, features known from completely integrable systems.

A.1.4. On Finite Energy Field Solitons Traveling at Speeds Less than Light

Since the source-free Maxwell–Born–Infeld field equations feature infinite-energy solutions with planar symmetry that display nonlinear soliton-like dynamics, naturally one wonders whether stable *finite-energy* solutions exist that could belong in the category particle-like soliton. In the following we provide some negative answers the proofs of which are

[39] Meanwhile, the complete integrability of the dynamical equations for plane simply periodic waves was proven in ref. 24. Interestingly, the equations are globally well-posed if and only if the data satisfy a smallness condition.

very different from the well-known proof of the "no-solitons theorem" of Derrick.

Since any soliton which travels at a speed less than the speed of light can always be boosted into a Lorentz frame in which it is at rest, the question of the existence of such solitons can be reduced to the question whether static finite-energy solutions of the source-free Maxwell–Born–Infeld solutions exist. Such an inquiry was carried out by Y. Yang,[(107, 108)] though only for Born's first field model[(16, 17, 19)] in which the $O(\beta^8)$ terms under the square root are missing from the Lagrangian. Yang[(107, 108)] showed by Moser's application of Harnack's inequality[(79)] that the only static electromagnetic entire solutions vanishing at infinity as $O(1/|s|)$ are $\boldsymbol{B} = 0$ *and* $\boldsymbol{D} = 0$. Yang also showed, by entirely different arguments which are rather similar to the proofs of the geometric Bernstein type theorems on Minkowski spacetime[40] by Calabi[(25)] and by Cheng and Yau,[(27)] that the only entire solutions of the source-free Maxwell–Born–Infeld field equations having finite energy which are electrostatic are given by $\boldsymbol{D} = 0$, and those which are magnetostatic are given by $\boldsymbol{B} = 0$. Since for purely electrostatic or purely magnetostatic solutions the field equations of Born's first model coincide with the Maxwell–Born–Infeld field equations, we can take over Yang's Bernstein results.

Proposition A.1. Let $(\boldsymbol{B}, \boldsymbol{D})$ be a static electromagnetic entire solution of the source-free Maxwell–Born–Infeld field equations for $\beta < \infty$. If $\mathscr{H}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) < \infty$ (finite energy), then if $\boldsymbol{B} = 0$ one also has $\boldsymbol{D} = 0$, and vice versa.

Unfortunately, Yang's Harnack–Moser result does not apply to the Maxwell–Born–Infeld field equations. However, we found that under an additional smallness condition, Moser's application of Harnack's inequality[(79)] can be adapted to the Maxwell–Born–Infeld field equations. Thus we have

Proposition A.2. Let $(\boldsymbol{B}, \boldsymbol{D})$ be a static electromagnetic entire solution of the source-free Maxwell–Born–Infeld field equations for $\beta < \infty$, which decay to zero as $|s| \to \infty$, say like $O(1/|s|)$, which implies $\mathscr{H}_{\text{field}}(\boldsymbol{B}, \boldsymbol{D}) < \infty$ (finite energy). Assume that there is a (positive) $\epsilon \ll 1$ such that the associated fields $(\boldsymbol{E}, \boldsymbol{H})$ satisfy the bounds $\beta^4\, |\boldsymbol{E}|^2 \leqslant 1 - \epsilon$ and $\beta^4\, |\boldsymbol{H}|^2 \leqslant 1 - \epsilon$, uniformly on $\mathbb{R}^3$. Then $\boldsymbol{B} = 0$ and $\boldsymbol{D} = 0$.

Sketch of Proof of Proposition A.2. We first note that from the Born and Infeld aether laws (30) and (31) it follows that when $(\boldsymbol{B}, \boldsymbol{D})$ go to

[40] This uses the same differential-geometric analogy which figured also in our discussion on static solutions with point sources alias maximal space-like slices in Minkowski spacetime.

zero at infinity, then so do $(\boldsymbol{E}, \boldsymbol{H})$, at the same rate. We next invert (30) and (31) to expresses $\boldsymbol{B}$ and $\boldsymbol{D}$ explicitly in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$,

$$\boldsymbol{B} = \frac{\boldsymbol{H} + \beta^4 \boldsymbol{E} \times (\boldsymbol{E} \times \boldsymbol{H})}{\sqrt{1 - \beta^4(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2) + \beta^8 \, |\boldsymbol{E} \times \boldsymbol{H}|^2}} \tag{127}$$

$$\boldsymbol{D} = \frac{\boldsymbol{E} + \beta^4 \boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{E})}{\sqrt{1 - \beta^4(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2) + \beta^8 \, |\boldsymbol{E} \times \boldsymbol{H}|^2}}, \tag{128}$$

for $\beta \in (0, \infty)$. The fields $\boldsymbol{B}$ and $\boldsymbol{D}$ must satisfy the vanishing divergence equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{D} = 0$, and since $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ and $\nabla \times \boldsymbol{H} = \boldsymbol{0}$ implies that there exist scalar fields f and g such that $\boldsymbol{E} = \nabla f$ and $\boldsymbol{H} = \nabla g$, the problem of finding entire static electromagnetic solutions of the Maxwell–Born–Infeld field equations reduces to solving a coupled system of two scalar PDEs of divergence form, namely

$$\nabla \cdot \left(\frac{\nabla g + \beta^4 \, \nabla f \times (\nabla f \times \nabla g)}{\sqrt{1 - \beta^4(|\nabla f|^2 + |\nabla g|^2) + \beta^8 \, |\nabla f \times \nabla g|^2}} \right) = 0, \tag{129}$$

and the same equation with f and g interchanged. The numerator between the big parentheses can be rewritten as

$$\nabla g + \beta^4 \, \nabla f \times (\nabla f \times \nabla g) = ((1 - \beta^4 \, |\nabla f|^2) \, \mathrm{Id} + \beta^4 \, \nabla f \otimes \nabla f) \cdot \nabla g, \tag{130}$$

so (129) reads $\nabla \cdot (M \cdot \nabla g) = 0$, with $M(\nabla f, \nabla g)$ a symmetric matrix. The eigenvalues of M are easily seen to be $m_1 = \beta^4 / \sqrt{\dots}$ and $m_2 = (1 - \beta^4 \, |\nabla f|^2) / \sqrt{\dots}$, where $\sqrt{\dots}$ denotes the denominator between the big parentheses in (129). Clearly, $\sqrt{\dots} \leqslant \sqrt{1 + \beta^8 \, |\nabla f \times \nabla g|^2}$, and for entire solutions for which $(\nabla f, \nabla g) \to (\boldsymbol{0}, \boldsymbol{0})$ at infinity, we have $\sqrt{1 + \beta^8 \, |\nabla f \times \nabla g|^2} < C$ (C a generic, positive constant), hence $m_1 \geqslant C > 0$ uniformly on $\mathbb{R}^3$. The same estimate for $\sqrt{\dots}$ combined with the bound $1 - \beta^4 \, |\nabla f|^2 \geqslant \epsilon$, which holds by hypothesis, shows that also $m_2 \geqslant C > 0$ uniformly on $\mathbb{R}^3$. Thus, (129) is strictly elliptic, and M symmetric, hence by Moser's application of Harnack's inequality[(79)] we conclude that $\nabla g = \boldsymbol{0}$. The same argument holds for (129)'s twin equation, hence also $\nabla f = \boldsymbol{0}$. ▮

Thus, nontrivial static electromagnetic entire solutions with finite-energy with both $\boldsymbol{B} \neq \boldsymbol{0}$ and $\boldsymbol{D} \neq \boldsymbol{0}$ can at most exist if the magnitude of the associated fields $\boldsymbol{E}$ and $\boldsymbol{H}$ exceeds a certain value. The possibility of finite-energy solitons which travel precisely at the speed of light is another open question.

A.2. The Action Principle

Any decent relativistic physical theory satisfies an action principle.[29] The geometrical equations of our classical theory are no exception. The action $\mathscr{A}$ is defined as integral over a four-dimensional spacetime domain Ξ sandwiched between two disjoint space-like slices, called the past and future boundaries of Ξ, thus $\mathscr{A} = \int_\Xi L$, where L is a four-form, called Lagrangian "density." The quotes here indicate that the point particle terms are not true densities but Dirac measures; having pointed this out we will from now on simply speak of Lagrangian density. We remark that the Dirac measure-valued four-forms can be recast as regular one-forms to be integrated along those truncated histories which are cut off by the future and past boundaries of Ξ, but to work out the Euler–Lagrange equations one then has to convert back to four-forms. In any event, the Lagrangian density naturally splits into a sum of two terms, one associated with the time-like line defects H_k of the electromagnetic potential, the other with the differentiable part that gives the electromagnetic field $\mathbf{F}$ of the electromagnetic spacetime between those defects, which gives the familiar

$$L = L_{\text{particle}} + L_{\text{field}}. \tag{131}$$

A.2.1. The Lagrangian Density in A and u_k Variables

The term L_{field} is the Born–Infeld Lagrangian density, which involves only the electromagnetic curvature $\mathbf{dA} = \mathbf{F}$ in $\Xi \setminus \bigcup_k \mathrm{H}_k$ (recall that $\mathbf{F}$ is defined by $\mathbf{dA} = \mathbf{F}$),

$$L_{\text{field}}(\varpi) \equiv \frac{1}{4\pi}\,{}^{\star}\!\left(\frac{\alpha}{\beta^4} - \frac{\alpha}{\beta^4}\sqrt{\det_{\mathrm{g}}(\mathbf{g} + \beta^2\,\mathbf{dA}(\varpi))}\right)\prod_{k\in\mathcal{N}} \chi_{\Xi\setminus\mathrm{H}_k}(\varpi), \tag{132}$$

where $\det_{\mathrm{g}}$ means determinant w.r.t. the metric $\mathbf{g}$ (for rank-two tensors on $\mathbb{M}^4$ we have $\det_{\mathrm{g}} = -\det$). The determinant can be expanded as follows,

$$\det_{\mathrm{g}}(\mathbf{g} + \beta^2\,\mathbf{dA}) = 1 - \beta^4\,{}^{\star}(\mathbf{F}\wedge{}^{\star}\mathbf{F}) - \beta^8({}^{\star}(\mathbf{F}\wedge\mathbf{F}))^2. \tag{133}$$

In the limit of a weak electromagnetic curvature, the Born–Infeld Lagrangian density reduces to the familiar Lagrangian density of the vacuum Maxwell fields,[78]

$$L_{\text{field}} \sim -\frac{1}{8\pi}\,\alpha\,\mathbf{F}\wedge{}^{\star}\mathbf{F}\prod_k \chi_{\Xi\setminus\mathrm{H}_k} \qquad \text{(weak field limit).} \tag{134}$$

The Lagrangian density L_{particle} involves only the line defects of the electromagnetic connection $\mathbf{A}$. It consists of a sum of linear and quadratic terms in the $\mathbf{u}_k$s,

$$L_{\text{particle}}(\varpi) \equiv \alpha \mathbf{A} \wedge \sum_{k \in \mathcal{N}} \int_{-\infty}^{+\infty} z_k^\star \mathbf{u}_k(\tau)\, \delta_{\eta_k(\tau)}(\varpi)\, \mathrm{d}\tau$$
$$+ \tfrac{1}{2} \sum_{k \in \mathcal{N}} \int_{-\infty}^{+\infty} \mathbf{u}_k(\tau) \wedge {}^\star\mathbf{u}_k(\tau)\, \delta_{\eta_k(\tau)}(\varpi)\, \mathrm{d}\tau. \tag{135}$$

The term quadratic in the $\mathbf{u}_k$s is the four-form version of a familiar expression,[(99)] the proper-time integral along point histories.[41] The term linear in the $\mathbf{u}_k$s can be written more concisely as $\alpha \mathbf{A} \wedge \mathbf{J}$ and reveals itself as the familiar "minimal coupling" term recast as a four-form on $\mathbb{M}^4$, cf. ref. 99. Note that because of the Dirac δ function we can alternatively pull $\mathbf{A}$ under the τ-integral and switch to $\tilde{\mathbf{A}}$.

The *relativistic principle of "least" action* demands that $\mathcal{A}$ be extremal w.r.t. independent variations of $\mathbf{A}$ and the $\mathbf{u}_k$. The variations w.r.t. $\mathbf{A}$ are standard and yield the Maxwell–Born–Infeld field equations with point sources. Notice that α does not figure in these variations. The variations w.r.t. $\mathbf{u}_k$ are constrained by the fact that each $\mathbf{u}_k$ is a Minkowski-velocity co-vector, i.e., dual to the unit tangent vector at the respective history, and as such satisfying ${}^\star(\mathbf{u}_k \wedge {}^\star\mathbf{u}_k) = 1$. These constraints are taken into account by adding to the Lagrangian density the term

$$L^{\Phi}_{\text{particle}}(\varpi) = -\sum_{k \in \mathcal{N}} \int_{-\infty}^{+\infty} \mathbf{d}_k \tilde{\Phi}(\eta_1(\tau),..., \eta_N(\tau)) \wedge {}^\star\mathbf{u}_k(\tau)\, \delta_{\eta_k(\tau)}(\varpi)\, \mathrm{d}\tau. \tag{136}$$

Note that after integration over Ξ each summand indeed vanishes, independently of the gauge, by virtue of the same reason why each summand in (11) separately satisfies the law of charge conservation (10). Unconstrained variation of $\mathcal{A}$ w.r.t. $\mathbf{u}_k$ now gives the guiding laws (17). The constraint ${}^\star(\mathbf{u}_k \wedge {}^\star\mathbf{u}_k) = 1$ (equivalent to $\mathbf{u}_k \cdot \mathbf{u}_k = -1$) applied to the guiding law then gives Eq. (18) for each $\mathbf{d}_k\tilde{\Phi}$. We note that $\tilde{\Phi}$ plays a rôle close to a familiar Lagrange parameter, which is what it would be if $\mathbf{A}$ were of class C^1. Here $\tilde{\Phi}$ turns out to have some life of its own.

[41] Notice that the quadratic term in the $\mathbf{u}_k$s is the only term which is not proportional to α; hence, formally we can "switch off" electromagnetic influences on the point histories by letting $\alpha \downarrow 0$, retaining only the quadratic term in the $\mathbf{u}_k$s. We thereby obtain the familiar Lagrangian for non-interacting particles which satisfy Galileo's law of inertia, viz. geodesic motion in $\mathbb{M}^4$.

A.2.2. The Lagrangian Density in **A** and Φ Variables

We note that we may switch from the $\mathbf{u}_k$s to $\tilde{\Phi}$ as variational degrees of freedom. Namely, similarly to defining **F** via $\mathbf{F} = \mathbf{dA}$, we may simply *define* $\mathbf{u}_k$ via $\mathbf{u}_k = \mathbf{d}_k\tilde{\Phi} - z_k\alpha\tilde{\mathbf{A}}_k$ and rewrite the action principle with **A** and $\tilde{\Phi}$ as variables. However, different from the definition of **F**, which automatically implies the Faraday–Maxwell law $\mathbf{dF} = 0$, the definition of $\mathbf{u}_k$ does not automatically imply that $\mathbf{u}_k$ is a Minkowski velocity co-vector, so that this piece of information has to be incorporated for the variations. The total Lagrangian density then reads

$$L(\varpi) = \frac{1}{4\pi}\,{}^\star\!\left(\frac{\alpha}{\beta^4} - \frac{\alpha}{\beta^4}\sqrt{\det_{\mathrm{g}}(\mathbf{g} + \beta^2\,\mathbf{dA}(\varpi))}\right)\prod_{k\in\mathcal{N}}\chi_{\Xi\setminus\mathrm{H}_k}(\varpi)$$
$$-\frac{1}{2}\int_{-\infty}^{\infty}\sum_{k\in\mathcal{N}}\left((\mathbf{d}_k\tilde{\Phi} - z_k\alpha\tilde{\mathbf{A}}_k)\wedge{}^\star(\mathbf{d}_k\tilde{\Phi} - z_k\alpha\tilde{\mathbf{A}}_k) + {}^\star 1\right)\Big|_{\{\varpi_n = \eta_n(\tau)\}}\,\delta_{\eta_k(\tau)}(\varpi)\,\mathrm{d}\tau \tag{137}$$

The action principle inherits a change from $z_k \in \{-1, 1\}$ to $z_k \in \mathbb{Z}$; to incorporate the κ_k, replace $z_k \to \kappa_k z_k$ and $\sum_{k\in\mathcal{N}} \to \sum_{k\in\mathcal{N}}\kappa_k^{-1}$ in (135), (136), and (137).

A.3. Extensions to General-Relativistic Spacetimes

The formal extension of our classical special-relativistic electromagnetic theory with point charges to a general-relativistic electromagnetic theory with point charges in which spacetime is no longer flat but dynamical, is perfectly straightforward. All we need to do, besides allowing **g** to be a general Lorentz metric with signature $+2$ and interpreting all the p forms and covariant derivatives accordingly, is to add the gravitational Lagrangian density of the Einstein–Hilbert variational principle to L,

$$L = L_{\text{particle}} + L_{\text{field}} + L_{\text{spacetime}} \tag{138}$$

with

$$L_{\text{spacetime}} = \frac{1}{8\pi\gamma}\,{}^\star\mathrm{tr}_{\mathrm{g}}\,\mathbf{R} \tag{139}$$

where $\mathbf{R}$ is the Ricci curvature tensor of the metric **g**, and γ is given in (3). Beside **A** and $\mathbf{u}_k$, also **g** is now a variable for the principle of least action. Variation w.r.t. **g** yields the Einstein equations of spacetime with the electromagnetic energy-momentum tensor as source of the spacetime curvature. The rôle of γ is that of a coupling constant which calibrates the gravitational influence of the electromagnetism (and perhaps other non-spacetime sources) on the spacetime curvature.[42]

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[42] Formally, by letting $\gamma \downarrow 0+$, we can switch off the gravitational influence of electromagnetism on the spacetime curvature and obtain the Einstein equations for so-called vacuum spacetimes. Beside the passive, flat Minkowski spacetime, other global solutions of the vacuum Einstein equations exist which are truly dynamical; see ref. 30.

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